Essential Mathematics for Economists: The Freshman Foundation

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2026

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Notice to current students: If you find an error or typo,
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Preface

What this book is

This book is the first volume of a three-part series, and it has one job: to take you from high-school mathematics to the working toolkit you will need in intermediate microeconomics, macroeconomics, and econometrics. The chapters move from arithmetic foundations through powers and roots, algebra, functions and linear relationships, systems of equations, nonlinear models, and finally financial mathematics and summation. Every tool arrives with the economic intuition attached from the start—percentages become inflation rates, exponents become compound growth, linear functions become demand curves—rather than bolted on at the end.

Volume II, the *Intermediate Toolkit*, is forthcoming: single-variable calculus, partial derivatives, unconstrained and constrained optimization, integration, and the probability and statistics you need for econometrics. Volume III, *Essential Mathematics for Economists: The Graduate School Bridge*, carries the story to linear algebra, real analysis, and optimization. Together, the three volumes are meant to be a coherent mathematics spine for an economics student, from the first freshman lecture to the first graduate seminar.

How to use this book

Every chapter opens with a short explanation of why the mathematics matters in economics, then builds the tools through worked examples you should read with a pencil in hand. Each chapter ends with two landmarks. *Learning outcomes* tell you what you should be able to do when you finish. *End-of-Chapter Exercises* give you the workout—and full worked solutions to every exercise are collected at the back of the book.

A word of advice about those solutions: attempt each exercise before you look. Reading a solution feels like understanding, but it is not the same as producing one yourself. If you get stuck, peek at the first line of the solution, close the book, and finish it on your own.

The mathematics in this book is elementary, but fluency—doing it quickly and without sign errors—is what your later courses will assume.

And if you spot an error or a typo anywhere in these pages, tell us. There is a small bounty for extra credit waiting for the first student to report each one.

Prerequisites

High-school mathematics: comfort with arithmetic and basic algebra. Nothing more. Some early chapters may feel easy—good. They are building the reflexes, the notation, and the economic vocabulary that the later chapters, and the later volumes, will take for granted.

Chapter 1

Arithmetic Foundations for Economics

1.1 Why Arithmetic Matters in Economics

You can't do economics without numbers. Before you can say anything serious about inflation, unemployment, income inequality, productivity, or economic growth, you need to be able to do the arithmetic reliably. Arithmetic looks elementary, but a slip with fractions, percentages, decimals, or rounding can flip an economic conclusion. Official statistics, financial reports, and empirical research all run on precise numerical computation—and price indices, real and nominal variables, and percentages show up in your very first economics courses.

The tools in this chapter will follow you through your whole economics education. Percentages measure inflation, unemployment, tax rates, interest rates, growth rates, and rates of return. Fractions and ratios become productivity measures, consumption shares, and market shares. Rounding matters when you read statistical output or report an estimate, while real and nominal values are the backbone of macroeconomics and public policy. Even the most advanced models you meet later rest on the numerical habits you build here.

1.2 Number Systems

Before you start calculating, it helps to know what kinds of numbers you are working with.

1.2.1 Integers ($\mathbb{Z}$)

The set of integers consists of all positive and negative whole numbers, including zero.

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

You will use integers whenever you count things that can go up or down: gains and losses, debts and credits, changes in inventory.

1.2.2 Natural Numbers ($\mathbb{N}$)

Natural numbers are the counting numbers beginning with zero.

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$$

These are the right tool whenever negative values are impossible—the number of consumers or firms, say.

1.2.3 Rational Numbers ($\mathbb{Q}$)

A rational number can be written as a ratio of two integers:

$$\frac{a}{b}, \qquad b \neq 0.$$

Every integer is rational too—just give it denominator 1. For example, $-6 = -6/1$. Terminating decimals such as 2.75 and repeating decimals such as $0.333\ldots$ are also rational.

1.2.4 Irrational Numbers

You can't write an irrational number as a ratio of two integers. Its decimal representation neither terminates nor repeats. Examples include

$$\sqrt{2}, \qquad \sqrt{5}, \qquad \pi, \qquad e.$$

1.2.5 Real Numbers ($\mathbb{R}$)

Real numbers are everything on the real number line. Every real number is either rational or irrational.

They include:

- Integers
- Rational numbers, including fractions and terminating or repeating decimals
- Irrational numbers

Examples include

$$\frac{3}{4}, \quad 2.5, \quad \sqrt{2}, \quad \pi, \quad e$$

In economics, prices, income, GDP, and interest rates are all modeled as real numbers.

1.3 Order of Operations

When an expression mixes several operations, you have to evaluate them in a specific order—everyone has to agree on the rules, or the same expression would mean different things to different people.

A useful mnemonic is **PEMDAS** (or **BODMAS**, where "brackets" and "orders" correspond to parentheses and exponents):

1. Parentheses
2. Exponents

3. Multiplication and Division (from left to right)
4. Addition and Subtraction (from left to right)

1.3.1 Example

Let's walk through one. Evaluate

$$8 - \left(2 + 6 \times \frac{1}{3}\right) + 2^{3-2}.$$

Step 1: Parentheses

Inside the parentheses, multiplication takes priority:

$$6 \times \frac{1}{3} = 2, \qquad 2 + 2 = 4.$$

Step 2: Exponents

Evaluate the exponent first:

$$3 - 2 = 1, \qquad 2^1 = 2.$$

Step 3: Addition and Subtraction

The expression is now evaluated from left to right:

$$8 - 4 + 2 = 4 + 2 = \boxed{6}.$$

1.3.2 Left-to-Right Evaluation

Multiplication and division have equal priority.

For example, the following expression must be evaluated from left to right:

$$8 \div 4 \times 2 = (8 \div 4) \times 2 = 2 \times 2 = 4.$$

It's mathematically correct, but it's asking to be misread. When you write your own expressions, use parentheses to make your intended meaning explicit.

1.4 A More Involved Example

Evaluate

$$\frac{3 + (8 - 3^{4-2})}{5 - 2 \times 2}.$$

Numerator

Compute the exponent first and then proceed in order:

$$4 - 2 = 2, \qquad 3^2 = 9, \qquad 8 - 9 = -1, \qquad 3 + (-1) = 2.$$

Denominator

Compute multiplication first:

$$2 \times 2 = 4, \qquad 5 - 4 = 1, \qquad \frac{2}{1} = 2.$$

Breaking large expressions into smaller pieces cuts your error rate dramatically—make it a habit.

1.5 Multiplication Notation

Multiplication can be written in several equivalent ways:

$$a \times b = a \cdot b = ab.$$

So when you see two variables side by side, read it as multiplication.

Prefer the dot notation: the symbol $\times$ looks too much like the variable x.

1.5.1 Implicit Multiplication

Expressions such as $4x$ should be interpreted as a single product. Consequently,

$$\frac{8}{4x} = \frac{8}{4 \cdot x} = \frac{2}{x}, \qquad \frac{8}{4x} \neq \left(\frac{8}{4}\right) x.$$

Here's a mental trick: whenever a fraction bar appears, imagine invisible parentheses around both the numerator and the denominator.

1.6 The Minus Sign and Parentheses

This is where most sign mistakes are born: a minus sign before parentheses hits every term inside.

For example,

$$-(5 - 10) = (-1)(5 - 10), \qquad 2 - (5 - 10) = 2 - 5 + 10 = 7.$$

Multiplying by -1 changes the sign of every term.

1.7 Working with Fractions

1.7.1 Adding Fractions

You can only add fractions once they share a denominator.

Example: the least common denominator of 12 and 16 is 48, so

$$\frac{5}{12} + \frac{3}{16} = \frac{20}{48} + \frac{9}{48} = \frac{20+9}{48} = \frac{29}{48}.$$

1.7.2 Multiplying and Dividing Fractions

When you multiply fractions, cancel first and multiply second—it saves time and mistakes.

Example:

$$\frac{25}{12} \div \frac{25}{24} = \frac{25}{12} \times \frac{24}{25} = \frac{24}{12} = \boxed{2}.$$

Here 25 cancels with 25, and $24/12 = 2$.

Simplifying before multiplying usually saves time and reduces arithmetic errors.

1.7.3 Mixed Numbers

One convention worth adopting: don't write mixed numbers such as $3\frac{1}{2}$ in algebraic calculations, because they read like multiplication. Write either 3.5 or $\frac{7}{2}$ instead.

1.8 Ratios, Proportions, and Unit Conversions

A ratio compares two quantities by division. The ratio of a to b can be written as $a : b$ or $\frac{a}{b}$, where $b \neq 0$. The order matters. For example, if a firm produces 600 units using 30 labor-hours, output per labor-hour is

$$\frac{600 \text{ units}}{30 \text{ labor-hours}} = 20 \text{ units per labor-hour.}$$

A proportion states that two ratios are equal. Cross multiplication gives

$$\frac{a}{b} = \frac{c}{d} \quad \Longrightarrow \quad ad = bc, \qquad b, d \neq 0.$$

For example, if three notebooks cost \$7.50 and eight notebooks cost \$x at the same unit price, then

$$\frac{3}{7.50} = \frac{8}{x} \quad \Longrightarrow \quad 3x = 60 \quad \Longrightarrow \quad x = 20.$$

Unit conversions can be performed by multiplying by conversion factors equal to one. If a product costs 90 cents per pound, its cost per ton is

$$90\frac{\text{cents}}{\text{pound}} \times \frac{\$1}{100 \text{ cents}} \times \frac{2{,}000 \text{ pounds}}{1 \text{ ton}} = 1{,}800\frac{\text{dollars}}{\text{ton}}.$$

Writing the units explicitly helps ensure that unwanted units cancel and the desired unit remains.

Economists convert units constantly—not just pounds to tons, but dollars to euros, or nominal to real—because you can't compare prices, incomes, or output across countries and years until everything is measured in the same units.

1.9 Decimal Numbers

1.9.1 Addition and Subtraction

Line up the decimal points—always.

For example, $0.120 + 0.034$ becomes

$$\begin{array}{r} 0.120 \\ +0.034 \\ \hline 0.154 \end{array}$$

The decimal points remain vertically aligned throughout the calculation.

1.9.2 Multiplication

Ignore the decimal points initially. For example, multiplying 0.12 by 0.006 as whole numbers gives $12 \times 6 = 72$. Because the factors have two and three decimal places, respectively, the product must contain five decimal places:

$$0.12 \times 0.006 = 0.00072.$$

1.10 Scientific Notation

Scientific notation expresses very large or very small numbers in the form

$$a \times 10^n,$$

where $1 \leq |a| < 10$ and n is an integer.

For example,

$$4{,}500{,}000 = 4.5 \times 10^6, \qquad 0.00032 = 3.2 \times 10^{-4}.$$

To multiply numbers written in scientific notation, multiply the leading numbers and add the exponents. For example,

$$(2 \times 10^3)(3 \times 10^4) = 6 \times 10^7.$$

Scientific notation is useful when working with quantities such as national income, population, government debt, or very small probabilities. (US GDP was roughly 2.8×10^{13} dollars in 2024—try writing that out in full.)

1.11 Percentages

A percentage is just a fraction with denominator 100. For example,

$$60\% = \frac{60}{100} = 0.60, \qquad 60\% \text{ of } 200 = 0.60 \times 200 = \boxed{120}.$$

To calculate a percentage of a quantity, multiply.

You will meet percentages everywhere in economics: inflation rates, unemployment rates, taxes, interest rates, and growth rates.

1.11.1 Percentage Change

The percentage change from an initial value to a new value is

$$\text{Percentage Change} = \frac{\text{New Value} - \text{Initial Value}}{\text{Initial Value}} \times 100\%,$$

provided the initial value is not zero.

For example, if the price of a good rises from \$80 to \$92, then

$$\frac{92 - 80}{80} \times 100\% = 15\%.$$

The price increased by 15%.

1.11.2 Percentage Points

When two values are themselves percentages, their difference is measured in *percentage points*. Suppose the unemployment rate rises from 4% to 5.5%. The increase is

$$5.5\% - 4\% = 1.5 \text{ percentage points.}$$

Measured relative to the original unemployment rate, however, the percentage increase is

$$\frac{5.5 - 4}{4} \times 100\% = 37.5\%.$$

Thus, percentage points and percentage changes describe different comparisons—mixing them up is one of the most common quantitative mistakes in public debate.

1.11.3 Successive Percentage Changes

Successive percentage increases and decreases do not generally cancel. If a \$100 price rises by 10% and then falls by 10%, the final price is

$$100(1.10)(0.90) = 99.$$

The second change is calculated from \$110 rather than from the original \$100, so the final price is 1% below its initial value. Notice the trap: the two 10% moves look symmetric, but they apply to different bases.

1.12 Application: Index Numbers and Real vs. Nominal Values

Here's a payoff for all that percentage work: converting nominal economic variables into real variables using an index number, such as the Consumer Price Index (CPI).

When the price index is normalized to 100 in the base year, the formula for finding the real value of an economic variable (like wages or GDP) is:

$$\text{Real Value} = \frac{\text{Nominal Value}}{\text{Price Index}} \times 100$$

If a worker's nominal wage is $\$50,000$ in a year when the CPI is 125, their real wage (in base-year purchasing power) is:

$$\text{Real Wage} = \frac{50,000}{125} \times 100 = 400 \times 100 = 40,000.$$

Once you're comfortable with fractions and percentages, inflation adjustments like this take seconds.

1.13 Rounding

Here's the golden rule: don't round until the very end.

Small rounding errors can accumulate and produce significantly different final answers.

1.13.1 Example

Suppose $A = \frac{1000}{900} = 1.111111\ldots$ and you wish to compute $1100 - 1000A$. Rounding at different stages gives

$$A \approx 1.1 \implies 1100 - 1000(1.1) = 0, \qquad A \approx 1.11 \implies 1100 - 1000(1.11) = -10.$$

A seemingly small difference in rounding changes the final answer substantially.

So keep every digit you can through the middle of a calculation, and round only the final answer.

1.14 Rules for Rounding

For 1.4563, the fourth decimal digit is 3, so $1.4563 \approx 1.456$ to three decimal places. For 1.4568, the next digit is 8, so $1.4568 \approx 1.457$.

The general rule is simple:

- If the next digit is 0–4, leave the retained digit unchanged.
- If the next digit is 5–9, increase the retained digit by one.

For example,

$$1.285 \approx 1.29 \quad \text{(two decimal places)}, \qquad 1.285 \approx 1.3 \quad \text{(one decimal place)}.$$

Chapter Summary

By the end of this chapter, you should be able to:

- distinguish among natural numbers, integers, rational numbers, irrational numbers, and real numbers;
- apply the correct order of operations;
- correctly interpret mathematical notation involving multiplication and fractions;
- handle negative signs and parentheses without error;
- add, subtract, multiply, and divide fractions;
- work with ratios, proportions, and unit conversions;
- perform arithmetic with decimal numbers;
- express and calculate with numbers in scientific notation;

- convert between percentages, fractions, and decimals;
- calculate percentage changes and distinguish them from percentage-point changes;
- calculate successive percentage changes;
- apply correct rounding procedures while minimizing accumulated error.

Exercises

1. Classify each number as an integer, natural number, real number, rational number (fraction), or irrational number.

a. -6

b. 0

c. $\frac{9}{14}$

d. 2.75

e. $\sqrt{5}$

f. $-0.414141\ldots$

2. For each number, identify all categories that apply: Natural ($\mathbb{N}$), Integer ($\mathbb{Z}$), Rational, or Real ($\mathbb{R}$).

a. 12

b. π

c. $\frac{3}{4}$

d. -7

e. $\sqrt{5}$

3. Rewrite each expression in a clearer way, and then simplify if possible.

a. $\frac{10x}{5}$

b. $4x \times 3y$

c. $2x \times x \times 5$

4. Calculate:

a. $6^2 + 4 \times (10 - 6)$

b. $\frac{3^4}{6} \times 3 - 2^2$

c. $\frac{3^4}{6\times 3} - 2^2$

d. $(4 + (10 - 2^2) \times 3) + 59 - 2 \times 4$

5. Calculate:

a. $3 \times 5 - 3 \times 3 - 3$

b. $(2 + \sqrt{7-3})^2$

c. $\frac{16+3\times(4-4^2)}{25-(7\times 3)}$

d. $2^{3+1} + 4^{4-1} \times 5^{6-2\times 3} + \left(\frac{729}{9^2\times 3} + 1\right)$

e. $\frac{8^{\frac{42}{7}} - (2+3) \times \frac{(-9)^2}{3} - 1 \times 4^{2\times 2-2}}{7^{\frac{21}{7}} + ((6+5)\times 5+2)}$

6. Evaluate the following expressions.

a. $3 + 4 \times 2$

b. $\frac{3}{4} + \frac{1}{2}$

c. $2.5(4) - 1.2$

d. $\frac{18}{3+3}$

e. $1.5 + \frac{3}{5}$

7. Add or subtract the fractions. Simplify your answers.

a. $\frac{3}{8} + \frac{5}{12}$

b. $\frac{7}{9} - \frac{2}{15}$

c. $\frac{11}{18} + \frac{5}{24}$

d. $\frac{13}{20} - \frac{3}{8}$

e. $\frac{4}{7} + \frac{5}{14} - \frac{3}{21}$

8. Calculate and simplify.

a. $\frac{7}{15} \div \frac{14}{25}$

b. $\frac{9}{16} \div \frac{27}{40}$

c. $\frac{5}{18} \times \frac{36}{25}$

9. Convert the following fractions and decimals.

a. Convert $\frac{3}{8}$ to a decimal.

b. Convert 0.75 to a fraction in lowest terms.

c. Convert $\frac{7}{20}$ to a decimal.

d. Convert 1.25 to a fraction in lowest terms.

e. Convert $\frac{9}{4}$ to a decimal.

10. Calculate:

a. $0.47 + 0.083$

b. $3.6 + 0.045 + 0.209$

c. $12.08 + 0.7 + 0.056$

d. $0.009 + 0.031 + 0.6$

11. Perform the following decimal multiplications. Show all your work and give your answers in decimal form.

a. 2.4×3.5

b. 0.8×6.2

c. 1.25×4.8

d. 0.35×0.4

e. 12.5×0.24

12. Round the following numbers:

a. Round 3.768 to the nearest tenth.

b. Round 12.43 to the nearest whole number.

c. Round 0.857 to the nearest hundredth.

d. Round 45.149 to the nearest tenth.

e. Round 8.995 to the nearest hundredth.

13. This exercise shows why rounding too early can create inaccurate answers. A store increases a price by 18%. The original price is $275.

a. Find the exact new price.

b. Suppose someone rounds 18% to 20%. What new price do they get?

c. What is the difference between the two answers?

14. Answer the following questions involving ratios, proportions, and unit conversions.

a. A household spends $750 of its $1,000 monthly budget on consumption. Find the ratio of consumption to the total budget.

b. A factory produces 480 units in 24 labor-hours. Find output per labor-hour.

c. Five identical notebooks cost \$12.50. Use a proportion to find the cost of eight notebooks.

d. A commodity costs 80 cents per pound. Find its cost in dollars per ton, using 2,000 pounds per ton.

15. Calculate the requested changes.

a. GDP increases from \$500 billion to \$540 billion. Find the percentage change.

b. A price falls from \$250 to \$225. Find the percentage change.

c. The unemployment rate rises from 4% to 5%. Find both the percentage-point increase and the percentage increase relative to the original rate.

d. A tax rate rises from 15% to 18%. Find both the percentage-point increase and the percentage increase relative to the original rate.

16. Calculate each successive percentage change.

a. A \$200 price rises by 15% and then falls by 15%. Find the final price and the overall percentage change from the original price.

b. A \$50,000 salary rises by 4% in each of two consecutive years. Find the salary after two years and the total percentage increase.

17. Work with scientific notation.

a. Write 7,250,000 in scientific notation.

b. Write 4.8×10^{-5} in ordinary decimal notation.

c. Calculate $(3 \times 10^4)(2 \times 10^3)$ and express the answer in scientific notation.

d. A country's GDP is 2.5×10^{12} dollars and its population is 5×10^7. Find GDP per person.

Chapter 2

Powers, Roots, and Logarithms

2.1 Why Powers, Roots, and Logarithms Matter in Economics

A lot of what economists study grows proportionally rather than linearly. Population growth, inflation, compound interest, technological progress, depreciation, and economic growth all evolve multiplicatively through time, making exponents and logarithms the right mathematical tools. This chapter introduces the algebra you need to describe these processes, and it sets up the mathematical language you will use throughout economics and finance.

You will see economists take logarithms of variables before estimating statistical models. The reason is practical: a log specification turns a multiplicative relationship into a linear one, and the estimated coefficients then read off as elasticities. Exponential and logarithmic functions also show up in Cobb-Douglas production functions, growth theory, asset pricing, and macroeconomic forecasting. What you learn here supports your later work in finance, macroeconomics, and econometrics, where log transformations are routine in empirical research.

2.2 Powers and Exponents

The expression

$$x^y$$

when y is a positive integer means that the base x is multiplied by itself y times. The number y is called the *exponent* or *power*. Zero, negative, and fractional exponents extend this definition according to the rules below.

For example,

$$4^3 = 4 \cdot 4 \cdot 4 = 64.$$

On a keyboard, exponentiation is often written `x^y`.

2.2.1 Basic Rules for Powers

These four rules do most of the work in this chapter—make sure you can use them fluently.

1. Product of powers with the same base. If two powers have the same base, add the exponents:

$$a^m \cdot a^n = a^{m+n}.$$

For example,

$$7^a \cdot 7^b = 7^{a+b}.$$

2. Quotient of powers with the same base. If two powers have the same base and are divided, subtract the exponents:

$$\frac{a^m}{a^n} = a^{m-n} \qquad (a \neq 0).$$

For example,

$$\frac{5^a}{5^b} = 5^{a-b}.$$

3. Power of a product. If the exponent is the same, multiply the bases first:

$$a^n b^n = (ab)^n.$$

Thus,

$$7^a \cdot 5^a = 35^a.$$

4. Power of a power. When a power is raised to another power, multiply the exponents:

$$(a^m)^n = a^{mn}.$$

So,

$$(3^a)^b = 3^{ab}.$$

All four follow directly from what "repeated multiplication" means. If one ever slips your mind, write out the multiplications and re-derive it.

Why should you care? Because economic growth is multiplicative. If GDP grows 3% a year for 10 years, it isn't 30% bigger—it's $(1.03)^{10} \approx 1.34$, about 34% bigger. Every one of those multiplications leans on the product rule, and Chapter 8 will show you where this idea leads: compound interest.

2.2.2 Special Cases

A few special cases you'll use constantly.

$$1^a = 1 \qquad \text{for any real } a.$$

For any nonzero number a,

$$a^0 = 1.$$

For $a \neq 0$,

$$a^{-m} = \frac{1}{a^m}.$$

For example,

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8}.$$

The expression 0^0 is undefined. One reason is that the exponent rules break down there. For instance, if you tried to apply the quotient rule to $0^a/0^a$, you'd be forced to interpret $0/0$, which is undefined.

2.3 Roots

Roots can be written as fractional powers. One restriction when working with real numbers: an even root requires a nonnegative argument, while an odd root may have a negative argument.

The square root of a number a is written

$$\sqrt{a} = a^{1/2}.$$

More generally, the nth root is

$$\sqrt[n]{a} = a^{1/n}.$$

For example,

$$4^{1/2} = \sqrt{4} = 2.$$

Since $4 = 2^2$,

$$4^{1/2} = (2^2)^{1/2} = 2^{2\cdot 1/2} = 2.$$

2.3.1 Square Roots of Squares

If a is a real number, then

$$\sqrt{a^2} = |a|,$$

not simply a. The absolute value is needed because square roots are defined to be nonnegative. Forgetting it is a common source of sign errors.

For example,

$$\sqrt{(-3)^2} = 3,$$

and also

$$\sqrt{3^2} = 3.$$

The absolute value of a number is defined by

$$|a| = \begin{cases} a, & a \geq 0, \\ -a, & a < 0. \end{cases}$$

Thus,

$$|-3| = 3, \qquad |17| = 17.$$

2.3.2 Even Powers and Multiple Solutions

If

$$x^4 = 81,$$

then both $x = 3$ and $x = -3$ are solutions, because

$$3^4 = 81 \qquad \text{and} \qquad (-3)^4 = 81.$$

This happens because an even power removes the sign of a negative number.

In general, equations with even powers may have two real solutions, while odd powers preserve the sign.

2.4 Logarithms

A logarithm answers one question: *to what power must I raise the base to obtain a given number?*

For $a > 0$, $a \neq 1$, and $b > 0$, the logarithm base a of b is written

$$\log_a b = x,$$

which means

$$a^x = b.$$

The number x is the logarithm.

For example,

$$\log_3 81 = 4,$$

because

$$3^4 = 81.$$

Similarly,

$$\log_{10} 1{,}000{,}000 = 6,$$

because

$$10^6 = 1{,}000{,}000.$$

2.4.1 Common Bases

Three logarithm bases appear frequently:

- base 2, especially in binary and computer-related settings;
- base 10, the common logarithm;
- base e, the natural logarithm.

The natural logarithm is written

$$\ln x,$$

which means

$$\ln x = \log_e x.$$

The number $e \approx 2.71828$ is an irrational number, and it turns out to be the natural base for anything that grows continuously.

When no base is written, the intended base depends on context. In economics, ln usually means base e.

2.5 Logarithm Rules

Logs are a machine for turning multiplication into addition and division into subtraction—that's their whole point.

2.5.1 Product Rule

If $a > 0$, $a \neq 1$, and $b, c > 0$, then

$$\log_a(bc) = \log_a b + \log_a c.$$

This rule is especially useful when a model contains products of variables.

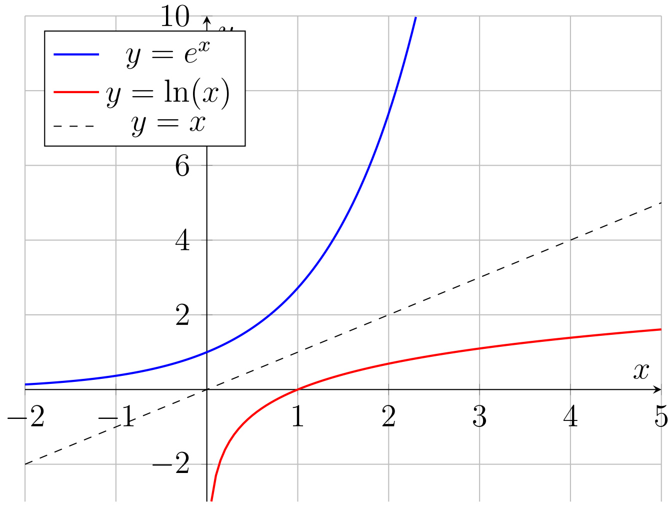


Figure 2.1: The natural exponential function and the natural logarithm function are reflections of each other across the line $y = x$.

2.5.2 Quotient Rule

Under the same restrictions,

$$\log_a\left(\frac{b}{c}\right) = \log_a b - \log_a c.$$

2.5.3 Power Rule

For $a > 0$, $a \neq 1$, and $b > 0$, a power inside a logarithm can be brought outside as a multiplier:

$$\log_a(b^k) = k \log_a b.$$

For example,

$$\ln(2^4) - \ln(8^2)$$

can be rewritten as

$$\ln\left(\frac{2^4}{8^2}\right).$$

Since $8 = 2^3$,

$$\frac{2^4}{8^2} = \frac{2^4}{(2^3)^2} = \frac{2^4}{2^6} = 2^{-2} = \frac{1}{4}.$$

Therefore,

$$\ln(2^4) - \ln(8^2) = \ln\left(\frac{1}{4}\right) = -2\ln 2.$$

2.6 Change of Base Formula

Sometimes a logarithm is awkward to evaluate in one base but easier in another. The change of base formula is

$$\log_a b = \frac{\log_c b}{\log_c a},$$

where c is any positive base other than 1.

Two common choices are $c = 10$ and $c = e$:

$$\log_a b = \frac{\ln b}{\ln a}.$$

This formula is especially useful when several different bases appear in the same expression.

2.6.1 Example

Here's a case where the formula earns its keep: a long product of logarithms in different bases that collapses, through cancellation, to a single number. Consider

$$\log_2 9 \cdot \log_3 16 \cdot \log_4 3 \cdot \log_9 4.$$

Using the change of base formula,

$$\log_2 9 = \frac{\ln 9}{\ln 2}, \qquad \log_3 16 = \frac{\ln 16}{\ln 3}, \qquad \log_4 3 = \frac{\ln 3}{\ln 4}, \qquad \log_9 4 = \frac{\ln 4}{\ln 9}.$$

Multiplying gives

$$\frac{\ln 9}{\ln 2} \cdot \frac{\ln 16}{\ln 3} \cdot \frac{\ln 3}{\ln 4} \cdot \frac{\ln 4}{\ln 9}.$$

Now most terms cancel:

$$= \frac{\ln 16}{\ln 2}.$$

Since $16 = 2^4$,

$$\ln 16 = \ln(2^4) = 4 \ln 2.$$

Hence,

$$\frac{\ln 16}{\ln 2} = 4.$$

So the entire expression equals

$$\boxed{4}.$$

2.7 Why These Rules Matter in Economics

Exponentials and logarithms are not just abstract algebraic tools. They are used in:

- compound interest and growth models,
- inflation calculations,
- elasticity and percentage changes,
- utility and production functions,
- interpreting regression coefficients in log-linear models.

2.7.1 Elasticity and Log-Linear Models

A particularly useful application of logarithms in economics is converting multiplicative models into linear models. For a Cobb-Douglas utility or production function like $U = x^{\alpha} y^{\beta}$, taking the natural logarithm yields:

$$\ln U = \alpha \ln x + \beta \ln y$$

In econometrics, when variables are transformed by natural logarithms before a regression, the estimated coefficients (α and β) represent *elasticities*—the percentage change in the dependent variable given a 1% change in the independent variable.

Chapter Summary

By the end of this chapter, you should be able to:

- interpret exponents as repeated multiplication;
- use the rules for multiplying, dividing, and powering expressions with the same base;
- rewrite roots as fractional powers;
- understand why square roots of squares produce absolute values;
- define and compute logarithms;
- apply the product, quotient, and power rules for logarithms;

- use the change of base formula to evaluate or simplify logarithms.

These ideas form the algebraic language behind many economic models.

Exercises

1. Simplify and calculate:

a. $5^2 \cdot 5^4$

b. $\frac{7^5}{7^2}$

c. $(3^2)^4$

d. $\frac{(4^3)^2}{4^4}$

2. Simplify the following expressions using the properties of the powers.

a. $2^3 \times 2^4$

b. $\frac{5^6}{5^2}$

c. $(3^2)^3 \times 3^{-1}$

d. $\frac{(2^3 \times 2^{-1})^2}{2^4}$

e. $\frac{4^3 \times 8^{-2}}{2^{-1}}$

3. Evaluate the roots.

a. $\sqrt{196}$

b. $\sqrt[3]{512}$

c. $\sqrt[4]{625}$

d. $\sqrt{225x^2}$

e. $32^{\frac{3}{5}}$

f. $(125^2)^{\frac{1}{3}}$

4. Solve for x.

a. $x^2 = 169$

b. $x^3 = 343$

c. $x^4 = 81$

5. Simplify the following expressions.

a. $16^{\frac{1}{2}}$

b. $27^{\frac{1}{3}}$

c. $81^{\frac{3}{4}}$

d. $32^{\frac{2}{5}}$

e. $64^{-\frac{1}{2}}$

6. Evaluate the logarithms.

a. $\log_2(128)$

b. $\log_3(243)$

c. $\log_{10}(0.001)$

d. $\log_5(1)$

7. Use logarithm rules to simplify.

a. $\log_2(8) + \log_2(4)$

b. $\log_3(81) - \log_3(3)$

c. $2\log_5(25)$

d. $\log_{10}(1000) + \log_{10}(0.1)$

8. Determine the value of each logarithmic expression.

a. $\log_2(8)$

b. $\log_3(81)$

c. $\log_5\left(\frac{1}{25}\right)$

d. $\log_2(32) + \log_2(4)$

e. $\log_3(81) - \log_3(9)$

9. Use the change-of-base formula to simplify the following expressions.

a. $\log_2(8) - \log_4(16)$

b. $\log_9(27) + \log_3(9)$

c. $\log_8(4) + \log_4(2)$

d. $\log_{25}(125) - \log_5(25)$

e. $\log_4(64) + \log_8(2)$

10. Evaluate.

a. $\frac{(2^5 \cdot 2^3)^2}{2^{10}}$

b. $\frac{16^3}{4^2}$

c. $\log_2\left(\frac{64 \cdot 8}{4}\right)$

d. $\sqrt[3]{125}\log_5(625)$

11. Simplify the following expressions using the properties of powers, roots, and logarithms.

a. $2^3 \cdot 4^{\frac{1}{2}}$

b. $81^{\frac{1}{4}} \cdot 3^2$

c. $\log_2(32) + \log_2(4) - \log_2(8)$

d. $\log_3(81^{\frac{1}{2}})$

e. $\frac{16^{\frac{3}{4}}}{2^2} + \log_4(64)$

12. Economists often approximate growth rates using the natural logarithm: $\ln(1+g) \approx g$, where g is expressed as a decimal. For each value of g, compute both $\ln(1+g)$ and g, then determine whether the approximation is accurate.

a. $g = 0.50$

b. $g = 0.02$

c. $g = -0.50$

d. $g = 0.10$

e. $g = -0.01$

Chapter 3

Algebraic Foundations for Economic Analysis

3.1 Why Algebraic Techniques Matter in Economics

Economic models are full of expressions you'll need to clean up before they mean anything to you. Economists solve equations to pin down equilibrium values, solve inequalities to describe feasible choices or policy constraints, and work with quadratic relationships when they analyze optimization problems. This chapter develops the algebraic techniques you need to manipulate these models systematically and accurately.

These techniques only get more important as you go. In intermediate microeconomics and macroeconomics, firms maximize profit, consumers maximize utility, and policymakers evaluate alternative policies—all by solving equations and comparing outcomes. Comparative statics—the study of how economic outcomes respond to changes in parameters—also relies on algebraic manipulation and monotonicity arguments. Even when later courses introduce calculus, you'll still be simplifying expressions, solving equations, and interpreting inequalities.

3.2 Simplifying Algebraic Expressions

Economic models frequently contain expressions with several variables. Your first move: combine like terms—and leave unlike terms alone.

For example,

$$8q + (10q - 6) - 15q = 3q - 6.$$

Only the coefficients of the common variable q are added. The constant term sits this one out.

With multiple variables, sort terms by type first (e.g. x^2, y^2, xy, x, y, constants)—expand carefully using the distributive law, then collect like terms.

3.3 Useful Algebraic Identities

You'll meet these three identities everywhere—optimization, comparative statics, all of it. Worth memorizing:

$$(x+y)^2 = x^2 + 2xy + y^2,$$

$$(x-y)^2 = x^2 - 2xy + y^2,$$

$$(x+y)(x-y) = x^2 - y^2.$$

3.4 Factoring Algebraic Expressions

Factoring runs expansion in reverse: turn a sum back into a product. Start by looking for a common factor. For example,

$$6x^2 + 9x = 3x(2x+3).$$

The difference-of-squares identity can also be read in reverse:

$$x^2 - y^2 = (x+y)(x-y).$$

Thus,

$$Q^2 - 64 = (Q+8)(Q-8).$$

Some quadratic expressions can be factored by finding two numbers whose product is the constant term and whose sum is the coefficient on the linear term. For example,

$$x^2 + 5x + 6 = (x+2)(x+3).$$

Factoring is the key to solving equations: a product equals zero exactly when at least one of its factors equals zero.

3.5 Solving Linear Equations

Solving an equation means isolating the unknown—do the same thing to both sides until it stands alone.

Consider the demand equation

$$Q = 120 - 2P + 8I.$$

If $Q = 500$ and $I = 80$,

$$500 = 120 - 2P + 8(80) = 760 - 2P,$$

which implies

$$-260 = -2P,$$

and therefore

$$P = 130.$$

Economically, that's the price consistent with the quantity demanded and income you observed.

3.6 Quadratic Profit Functions and Break-Even Points

Not everything is linear—profit functions are often quadratic (parabolas). A quadratic equation is written as:

$$ax^2 + bx + c = 0 \quad (a \neq 0)$$

The solutions (roots) of this equation are given by the **Quadratic Formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If a monopoly firm faces a downward-sloping linear demand curve $P = 100 - Q$, its revenue is quadratic: $R(Q) = P \cdot Q = (100 - Q)Q = 100Q - Q^2$. If total cost is $C(Q) = 20Q$, profit is:

$$\pi(Q) = R(Q) - C(Q) = -Q^2 + 80Q$$

To find the break-even points where profit equals zero, solve:

$$-Q^2 + 80Q = 0 \implies Q(-Q + 80) = 0$$

The firm breaks even at $Q = 0$ and $Q = 80$.

The vertex of a quadratic function $ax^2 + bx + c$ occurs at

$$x = -\frac{b}{2a}.$$

Because the coefficient on Q^2 in the profit function is negative, the parabola opens downward and its vertex gives the maximum profit. Here,

$$Q^* = -\frac{80}{2(-1)} = 40,$$

and

$$\pi(40) = -(40)^2 + 80(40) = 1600.$$

Thus, the profit-maximizing output is 40 units and the maximum profit is 1600.

3.7 Solving Inequalities

Inequalities work like equations, with one famous exception: multiplying or dividing by a negative number flips the inequality sign.

For example,

$$500 - 2Q > 60 + 3Q$$

implies

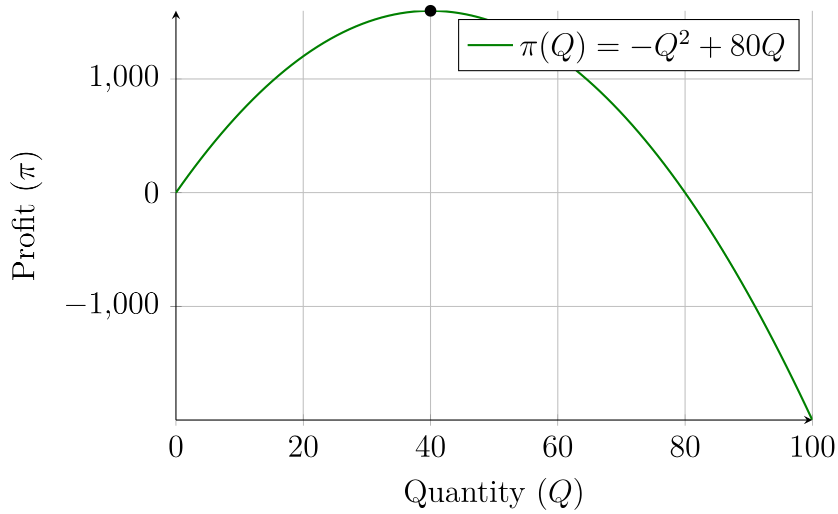


Figure 3.1: A quadratic profit function. The vertex of the parabola represents the profit-maximizing output level.

$$440 > 5Q,$$

so that

$$Q < 88.$$

That inequality describes every Q that works—the whole solution set.

3.8 Monotonicity and Comparative Statics

Economists constantly ask: when this variable goes up, does that one go up or down?

For example,

$$\frac{100 - x}{x^2 + 7}$$

decreases as x increases for $0 < x \leq 100$ because the nonnegative numerator decreases while the positive denominator increases.

Here's the canonical economic example of the same logic: average fixed cost, F/Q. As output Q grows, the numerator is fixed while the denominator grows, so average fixed cost falls—you're spreading the fixed cost over more units. You'll use this argument constantly.

When both numerator and denominator move in the same direction, additional algebra may be required before the direction of change becomes clear.

3.9 Absolute Values

The absolute value of a number measures its distance from zero:

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

To solve

$$|2x + 3| = 17,$$

consider both possibilities:

$$2x + 3 = 17$$

or

$$2x + 3 = -17.$$

Hence,

$$x = 7$$

or

$$x = -10.$$

3.10 Solution Sets

All the values that satisfy an equation or inequality make up its *solution set*. Sometimes there's exactly one, sometimes several, sometimes infinitely many.

Chapter Summary

By the end of this chapter, you should be able to:

- simplify algebraic expressions by expanding and collecting like terms;
- apply common algebraic identities;
- factor expressions by removing common factors and recognizing common quadratic patterns;
- solve linear equations for an unknown variable;
- analyze quadratic profit functions and find break-even points;
- solve inequalities and reverse the inequality sign when multiplying or dividing by a negative number;
- determine whether expressions increase or decrease over a stated domain;
- rewrite and solve expressions involving absolute values;
- describe the solution set of an equation or inequality.

The techniques presented in this chapter provide the algebraic foundation required for later topics in economics, including optimization, comparative statics, consumer theory, and producer theory.

Exercises

1. Simplify the following expressions.

a. $3x + 5x - 2x$

b. $4y - 3 + 2y + 7$

c. $2a + 3b - a + 5b$

d. $5(2x - 3) - 2(x + 4)$

e. $3(2x + y) - 2(x - 4y)$

f. $6m^2 - 2m + 3m^2 + 7m$

g. $12q + 7q - 4(3q + 2)$

2. Expand and simplify.

a. $(3 - x)^2$

b. $(y + 5)^2$

c. $(x - 4)(x + 4)$

d. $(1 - 2x)^2 + (y - 3)^2 + x(y + 2) - 2xy$

3. Simplify.

a. $(x + 2)^2 - (x - 2)^2$

b. $(x + y)^2 - (x - y)^2$

4. Factor each expression completely.

a. $8x^2 + 12x$

b. $Q^2 - 49$

c. $x^2 + 7x + 12$

d. $2y^2 - 18$

5. Solve for the indicated variable.

a. Solve for x: $3x + 5 = 20$

b. Solve for p: $4p - 7 = 21$

c. Solve for y: $\frac{y}{5} + 2 = 8$

d. Solve for a: $2(a - 3) = 10$

e. Solve for q: $P = 100 - 4q$

6. A coffee shop estimates daily demand for iced coffee using the equation $q = 200 - 4p + 5I$ where q is the number of iced coffees sold per day, p is the price of one iced coffee, and I is a local income index. On a hot summer day, the shop sells 400 iced coffees, and the income index is $I = 60$. Find the price.

7. Solve the following inequalities.

a. $3x + 4 > 10$

b. $5p - 7 \leq 18$

c. $\frac{y}{4} + 3 < 8$

d. $2(a - 1) \geq 12$

e. $-3q + 6 > 15$

8. Determine whether each fraction increases, decreases, or remains unchanged when the variable increases over the stated domain.

a. $\frac{p^2+1}{7}$ for $p \geq 0$

b. $\frac{5}{\sqrt{q}+2}$ for $q \geq 0$

c. $\frac{\ln(r)+4}{6}$ for $r > 0$

d. $\frac{2s^2+3}{8}$ for $s \geq 0$

e. $\frac{5}{t^2+4}$ for $t > 0$

9. Determine whether the expression increases or decreases. Assume $x > 0$.

a. $\frac{5}{x}$

b. $2x + 7$

c. $-x + \frac{3}{x}$

d. $\frac{50-x}{x+10}$ $(0 < x < 50)$

10. Rewrite the following absolute value expressions without absolute value signs.

a. $|2x - 6|$

b. $|p + 4|$

c. $|5 - q|$

d. $|r^2 - 9|$

e. $|3s + 12|$

11. Solve the following absolute value equations.

a. $|x| = 12$

b. $|3x - 1| = 8$

12. Solve the following absolute value inequalities.

a. $|x - 3| < 5$

b. $|2p+1| \leq 7$

c. $|y-4| > 3$

d. $|3a-6| \geq 12$

e. $|q+2| < 4$

Chapter 4

Algebra and Economic Modeling

4.1 Why Algebra Matters in Economic Modeling

Economic models simplify complex real-world decisions so that their central relationships can be studied. Algebra translates economic stories into mathematical relationships: variables represent quantities such as prices, output, income, or costs, while equations describe how those variables interact. Once you've expressed an economic situation mathematically, you can evaluate alternative decisions, compare policies, and predict the consequences of changes in economic conditions. The potato production example, consumer expenditure example, and many of the end-of-chapter exercises illustrate this process.

This modeling approach is used across economics. Firms build cost, revenue, and profit functions to study production decisions; consumers use expenditure and utility models; governments analyze tax policies and public spending; and economists estimate demand, supply, and production functions from data. Later courses introduce more sophisticated techniques, but the ability to formulate an economic problem mathematically and interpret the resulting equations stays with you.

Consider the following example.

Suppose you plan to grow potatoes. Each pound of potatoes requires 0.1 square feet of land, and the rental price is \$3 per square foot. Seeds and fertilizer used under ordinary conditions cost \$0.10 per pound of potatoes. As production expands, however, the farmer must use less suitable land, work overtime, or use equipment more intensively. We represent these capacity pressures by an additional cost of $Q^2/500$ dollars when Q pounds are produced. In addition, you must pay a fixed fee of \$100 in order to sell your potatoes at the supermarket, regardless of how many pounds you produce.

The question is: how can total cost, revenue, and profit be expressed as functions of the number of pounds of potatoes produced?

4.2 Building the Cost Function

Let Q denote the number of pounds of potatoes produced.

Since each pound requires 0.1 square feet of land, producing Q pounds requires

$$0.1Q$$

square feet of land.

Because land costs \$3 per square foot, the land rental cost is

$$3(0.1Q) = 0.3Q.$$

4.2.1 Seed, Fertilizer, and Capacity Costs

At ordinary production levels, seeds and fertilizer cost \$0.10 per pound, so this part of cost is

$$0.1Q.$$

Producing more also creates capacity pressures: the farmer may need to cultivate less suitable land, pay overtime wages, or use equipment more intensively. We represent this additional cost by

$$\frac{Q^2}{500}.$$

Unlike a linear cost, this quadratic component grows more than proportionally. For example, doubling output makes this component four times as large.

4.2.2 Variable Cost

The total variable cost is the sum of land rental cost, ordinary seed and fertilizer cost, and capacity-pressure cost:

$$C_V(Q) = 0.3Q + 0.1Q + \frac{Q^2}{500} = 0.4Q + \frac{Q^2}{500}.$$

Notice that $C_V(Q)$ depends on the quantity Q, so it is a function of Q.

4.3 Fixed Cost and Total Cost

In addition to variable cost, there is a fixed cost of \$100 required to sell the potatoes at the supermarket. This cost does not depend on output, so we write

$$C_F = 100.$$

Therefore, the total cost function is

$$C(Q) = 100 + 0.4Q + \frac{Q^2}{500}.$$

4.4 Revenue and Profit

Now consider the benefit side of the problem. Suppose potatoes sell for \$2 per pound. If Q pounds are sold, then total revenue is

$$R(Q) = 2Q.$$

Profit is revenue minus cost, so

$$\pi(Q) = R(Q) - C(Q).$$

Substituting the expressions above gives

$$\pi(Q) = 2Q - \left(100 + 0.4Q + \frac{Q^2}{500}\right).$$

This simplifies to

$$\pi(Q) = -\frac{Q^2}{500} + 1.6Q - 100.$$

Completing the square gives

$$\pi(Q) = 220 - \frac{(Q-400)^2}{500}.$$

Because $(Q-400)^2$ cannot be negative, profit is largest when this squared term equals zero. Therefore,

$$\boxed{Q^* = 400 \text{ pounds}}, \qquad \boxed{\pi(Q^*) = \$220}.$$

The quadratic capacity-pressure term makes total cost eventually grow faster than the linear revenue function, so the profit-maximizing quantity is finite.

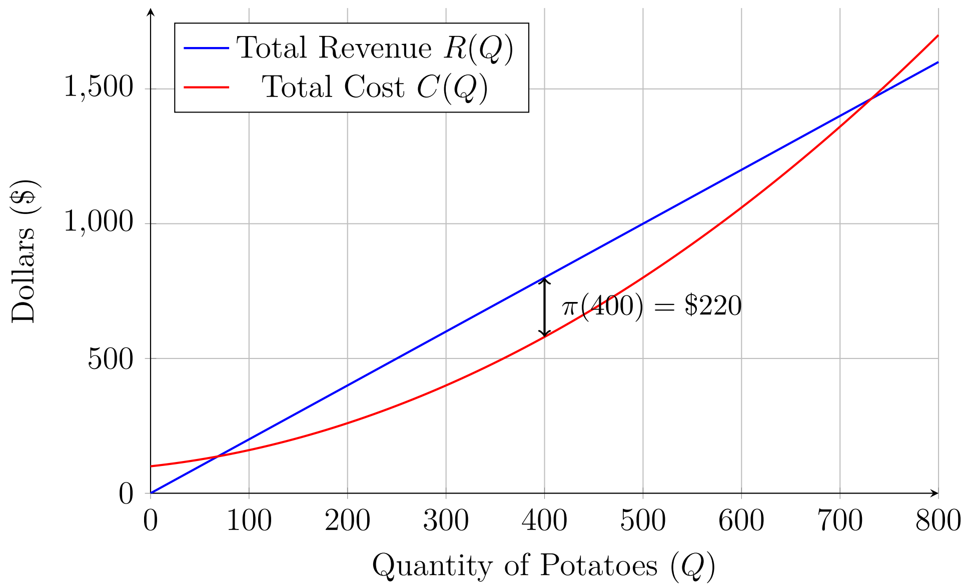


Figure 4.1: Total revenue and total cost functions for the potato farmer. Their intersections are break-even quantities. The vertical distance between revenue and cost is greatest at $Q = 400$, where profit is \$220.

4.5 Interpreting the Model

This example shows how an economic story can be translated into a mathematical model:

- The variable Q represents the quantity of potatoes produced.
- The cost function combines fixed cost, ordinary variable cost, and a quadratic capacity-pressure cost.
- The revenue function captures income from sales.
- The profit function measures net benefit.

One more thing: Q can't be negative here—not just wrong, meaningless. A negative quantity of potatoes lies outside the economically feasible domain, so exclude it.

Also, real economic models may involve more than one variable. For example, a farmer might choose how many pounds of potatoes and how many pounds of tomatoes to produce. In that case, the model would include multiple decision variables.

4.6 A Consumer Example

Algebra also models consumer choice.

Suppose a consumer buys apples and oranges. Let:

$$X = \text{number of apples}, \qquad Y = \text{number of oranges}.$$

If each apple costs \$2 and each orange costs \$3, then total spending is

$$E(X, Y) = 2X + 3Y.$$

For example, if the consumer buys 4 apples and 7 oranges, then

$$E(4, 7) = 2(4) + 3(7) = 8 + 21 = 29.$$

So the total payment is \$29.

Notice that $E(X, Y) = 2X + 3Y$ is exactly the left-hand side of the budget constraint you'll meet in Chapter 5. There, the same expression—expenditure on two goods—is set against income: $2X + 3Y \leq I$. The apples-and-oranges budget line is this expenditure function with a budget attached.

4.7 Evaluating the Potato Model

Evaluation means substituting particular values into a function.

4.7.1 Example 1: $Q = 200$

If $Q = 200$, then

$$C_V(200) = 0.4(200) + \frac{200^2}{500}.$$

Therefore,

$$C_V(200) = 80 + 80 = 160.$$

The total cost is

$$C(200) = 100 + 160 = 260.$$

Revenue is

$$R(200) = 2(200) = 400.$$

So profit is

$$\pi(200) = 400 - 260 = 140.$$

4.7.2 Example 2: $Q = 400$

If $Q = 400$, then

$$C_V(400) = 0.4(400) + \frac{400^2}{500}.$$

Therefore,

$$C_V(400) = 160 + 320 = 480.$$

The total cost is

$$C(400) = 100 + 480 = 580.$$

Revenue is

$$R(400) = 800.$$

So profit is

$$\pi(400) = 800 - 580 = 220.$$

4.7.3 Example 3: $Q = 600$

If $Q = 600$, then

$$C_V(600) = 0.4(600) + \frac{600^2}{500}.$$

Therefore,

$$C_V(600) = 240 + 720 = 960.$$

The total cost is

$$C(600) = 100 + 960 = 1060.$$

Revenue is

$$R(600) = 1200.$$

So profit is

$$\pi(600) = 1200 - 1060 = 140.$$

The quantities 200 and 600 are equally far from the profit-maximizing quantity 400, so the quadratic profit function gives the same profit, \$140, at both quantities.

4.8 Why This Matters in Economics

This kind of algebraic modeling is central to economics. Once a real-world situation is expressed mathematically, it becomes possible to:

- compare different choices,
- study how profit changes with output,
- identify the best production level,
- and analyze how prices and costs affect decisions.

Here the profit-maximizing output was found by completing the square. In intermediate microeconomics, the same choice would typically be analyzed by comparing marginal revenue and marginal cost or by using calculus.

Chapter Summary

By the end of this chapter, you should be able to:

- define the economic variables in a mathematical model;
- translate costs and revenues into functions;
- combine those functions to form profit and maximize a quadratic profit function;
- evaluate functions at specific values;
- interpret the resulting values economically.

Algebra gives economists a precise language for describing and analyzing real decision problems.

Exercises

1. A farmer grows carrots. Let q be pounds of carrots produced. Each pound requires 0.2 square feet of land. Land rent is \$2 per square foot. As production expands, increasingly intensive use of land, labor, and equipment creates a capacity-pressure cost of $q^2/500$ dollars. Fixed market fee is \$120.

a. Find the land rental cost.

b. Find the capacity-pressure cost.

c. Find $TVC(q)$.

d. Find $TC(q)$.

2. Use the result from Question 1. Evaluate $TC(q)$ when:

a. $q = 400$

b. $q = 900$

3. Use the conditions from Question 1. Suppose the carrots sell for \$3 per pound.

a. Find the revenue function $R(q)$.

b. Find the profit function $\pi(q)$.

c. Complete the square to find the profit-maximizing quantity and maximum profit.

4. A fruit stand sells apples and oranges. Suppose x apples and y oranges are sold. Apples cost \$2.50 each. Oranges cost \$1.80 each.

a. Write the total cost function $C(x, y)$.

b. Evaluate $C(12, 8)$.

c. Evaluate $C(20, 15)$.

5. A company's cost function is $C(q) = 500 + 2.5q + \frac{q^2}{200}$.

a. Find $C(100)$.

b. Find $C(400)$.

6. A firm's revenue function is $R(q) = 6q$ and its cost function is $C(q) = 300 + 3q + \frac{q^2}{200}$.

a. Write the profit function.

b. Find profit when $q = 100$.

c. Complete the square to find the profit-maximizing quantity and maximum profit.

7. A farm grows potatoes and tomatoes. Potatoes require 0.1 square feet per pound. Tomatoes require 0.15 square feet per pound. Land costs \$3 per square foot. Let $p =$ pounds of potatoes and $t =$ pounds of tomatoes.

a. Write the total land requirement.

b. Write the land rental cost.

c. Evaluate the rental cost when $p = 800$ and $t = 600$.

8. You are planning to produce handcrafted candles to sell at a local market. Each candle requires 0.5 pounds of wax, and wax costs \$4 per pound. The candles are packaged in boxes that hold 10 candles each, and each box costs \$3. Assume that q is a multiple of 10. In addition, you must pay a monthly booth rental fee of \$150 to sell your products. Suppose you plan to produce and sell $q \geq 0$ candles.

a. How much wax is required to produce q candles?

b. What is the total wax cost (in terms of q)?

c. What is the packaging cost (in terms of q)?

d. The sum of the wax and packaging costs is called the total variable cost, $TVC(q)$. Write an expression for $TVC(q)$.

e. Together with the booth rental fee of \$150 (the total fixed cost, TFC), what is the total cost of producing and selling q candles, $TC(q)$?

f. If each candle sells for \$8, what is the revenue function $R(q)$?

g. What is the profit function $\Pi(q) = R(q) - TC(q)$?

9. You operate a small bakery that sells muffins and cookies. Each muffin requires 0.2 pounds of flour and 0.1 pounds of sugar. Each cookie requires 0.1 pounds of flour and 0.05 pounds of sugar. Flour costs \$2 per pound and sugar costs \$1.50 per pound. In addition, the bakery pays a monthly rent of \$300. Suppose you produce and sell $m \geq 0$ muffins and $c \geq 0$ cookies.

a. How many pounds of flour are required to produce m muffins and c cookies?

b. How many pounds of sugar are required to produce m muffins and c cookies?

c. What is the total flour cost?

d. What is the total sugar cost?

e. Let $TVC(m, c)$ denote the total variable cost. Write an expression for $TVC(m, c)$.

f. Together with the monthly rent of \$300 (the total fixed cost, TFC), what is the total cost function $TC(m, c)$?

g. If muffins sell for \$4 each and cookies sell for \$2 each, what is the revenue function $R(m, c)$?

h. What is the profit function $\Pi(m, c) = R(m, c) - TC(m, c)$?

10. A movie theater sells tickets for adults and children. Let p_A denote the adult ticket price and p_C denote the child ticket price. Market research indicates that weekly demand decreases by 40 tickets for every \$1 increase in the adult ticket price and by 20 tickets for every \$1 increase in the child ticket price. If both ticket prices were \$0, the theater estimates that 1200 tickets would be demanded per week. Let $D(p_A, p_C)$ denote the weekly demand.

a. What is the effect of a \$1 increase in the adult ticket price on demand?

b. What is the effect of a \$1 increase in the child ticket price on demand?

c. Write an expression for the weekly demand function $D(p_A, p_C)$.

11. A country imports electric vehicles (EVs). Let p denote the price of an EV (in thousands of dollars). Economists estimate that when the price of an EV increases by \$1,000, monthly domestic demand decreases by 150 vehicles. If EVs were free ($p = 0$), consumers would demand $15,000$ vehicles per month.

a. Write the domestic demand function $D(p)$.

b. The government introduces a tariff of t thousand dollars per vehicle. If the tariff is added to the price paid by consumers, what price do consumers face?

c. Write the new domestic demand function $D(p, t)$.

d. Suppose the price of an EV is \$30,000 and the tariff is \$4,000. What is the resulting demand?

12. The international price of oil depends on the quantity supplied by three oil-producing countries: Petronia, Crudestan, and Barrelia. Let q_P, q_C, and q_B denote their daily production levels (in millions of barrels per day). Economists estimate that the price of oil (in dollars per barrel) is given by $P(q_P, q_C, q_B) = 120 - 2q_P - 3q_C - q_B$.

a. What is the price of oil if Petronia produces 10 million barrels per day, Crudestan produces 8 million barrels per day, and Barrelia produces 12 million barrels per day?

b. What is the price of oil if Petronia continues to produce 10 million barrels per day and Crudestan continues to produce 8 million barrels per day, but Barrelia stops producing oil ($q_B = 0$)?

13. A company buys a piece of equipment for \$24,000. The equipment loses \$2,500 in value each year.

a. Write a function $V(t)$ for the value of the equipment after t years.

b. What is the value of the equipment after 2 years?

c. What is the value of the equipment after 4 years?

d. What is the value of the equipment after 6 years?

e. After how many years will the equipment be worth \$11,500?

14. A financial analyst creates a simple AI stock index based on three companies: AlphaAI, NeuralNet, and RoboChip. The index is a weighted average of the three stock prices $I(a, n, r) = 0.5a + 0.3n + 0.2r$. In each case, use the given stock prices to evaluate the index.

a. The three stocks rise together in a calm market: $a = 40$, $n = 30$, and $r = 20$. What is the value of the index?

b. AlphaAI jumps while the other two stocks dip slightly: $a = 50$, $n = 28$, and $r = 18$. What is the value of the index?

c. NeuralNet is the main driver of a strong tech rally: $a = 36$, $n = 48$, and $r = 20$. What is the value of the index?

d. RoboChip drops sharply while the other two stocks remain fairly steady: $a = 42$, $n = 31$, and $r = 10$. What is the value of the index?

e. The whole sector moves up, but NeuralNet leads the way: $a = 45$, $n = 55$, and $r = 25$. What is the value of the index?

Chapter 5

Functions and Linear Relationships in Economics

5.1 Why Functions and Linear Relationships Matter in Economics

Functions are the heart of economics: they tell you how one variable depends on another. Demand depends on price, consumption depends on income, production depends on labor and capital, and tax revenue depends on economic activity. Reading a function and its graph means reading an economic relationship, not just a pile of numbers. This chapter introduces these ideas through linear functions, which provide simple and widely used approximations in economics.

Linear functions appear throughout economics because they are easy to interpret and often approximate more complicated relationships. The slope measures how one variable responds to changes in another, while intercepts often represent economically meaningful benchmark values. Budget constraints, demand curves, supply curves, consumption functions, and many introductory macroeconomic models are linear. Even when economists study nonlinear models, they frequently approximate them locally with linear relationships.

5.2 Functions

A function is a rule that assigns exactly one output to each allowable input—no input gets two answers. If the independent variable is denoted by x and the dependent variable by y, we write

$$y = f(x).$$

The set of allowable inputs is the *domain*, while the collection of attainable outputs is the *range*. Every input must correspond to one and only one output.

For example, consider

$$f(x) = \sqrt{x-2}.$$

The expression inside the square root must be nonnegative, so $x - 2 \geq 0$. Therefore, the domain is $x \geq 2$. Because the principal square root is nonnegative, the range is $f(x) \geq 0$.

Now consider

$$g(x) = \frac{10}{x-3}.$$

The denominator cannot equal zero, so the domain contains every real number except $x = 3$. The output can never equal zero, so the range contains every real number except $g(x) = 0$.

Economics often narrows the math: for the demand function

$$Q = 120 - 2P,$$

requiring both price and quantity to be nonnegative gives

$$0 \leq P \leq 60$$

and

$$0 \leq Q \leq 120.$$

5.3 Function Graphs

A function can be represented graphically on the Cartesian plane. Each point on the graph has coordinates (x, y), where $y = f(x)$. The graph is therefore the collection of all input–output pairs satisfying the function.

5.4 Linear Functions

A basic and widely used class of functions in economics is the linear function,

$$y = ax + b,$$

where a is the slope and b is the vertical intercept.

An example from macroeconomics is the consumption function

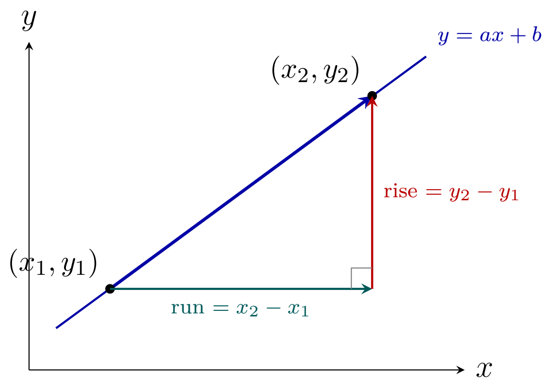
$$C = 0.6Y + 9000,$$

where Y denotes household income and C denotes consumption expenditure. The slope, 0.6, is the marginal propensity to consume (MPC): each extra dollar of income raises consumption by 60 cents. The intercept, 9000, is autonomous consumption—what households would spend even with zero income, financed out of savings or borrowing. Notice that this chapter just taught you to interpret intercepts, and here is one doing economic work.

5.5 Slope

Given two points (x_1, y_1) and (x_2, y_2), with $x_2 \neq x_1$, moving from the first point to the second gives a horizontal change (run) of $x_2 - x_1$ and a vertical change (rise) of $y_2 - y_1$. Thus the slope, or rise over run, is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$



$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Why the slope equals a.

Suppose both points lie on $y = ax + b$. Then

$$y_1 = ax_1 + b, \qquad y_2 = ax_2 + b.$$

Subtracting the first equation from the second gives

$$\begin{aligned} y_2 - y_1 &= (ax_2 + b) - (ax_1 + b) \\ &= a(x_2 - x_1). \end{aligned}$$

Because $x_2 \neq x_1$, division by $x_2 - x_1$ yields

$$\frac{y_2 - y_1}{x_2 - x_1} = a.$$

Thus every pair of distinct points on the line has slope a. The intercept b cancels: a vertical shift changes neither rise nor slope.

5.6 Demand Functions

A standard demand function is

$$Q = 120 - 2P,$$

where Q denotes quantity demanded and P denotes price.

Solving for price yields the inverse demand function,

$$P = 60 - \frac{Q}{2}.$$

Economists often graph inverse demand because price is conventionally placed on the vertical axis.

5.7 Alternative Forms of a Line

Linear equations can also be written as

$$Ax + By + C = 0.$$

Provided $B \neq 0$, solving for y produces

$$y = -\frac{A}{B}x - \frac{C}{B},$$

showing immediately that the slope equals $-A/B$.

5.8 Determining a Line

A unique straight line is determined by either

- two distinct points, or
- one point together with its slope.

Given two points, first compute the slope and then substitute one point into $y = ax + b$ to recover the intercept.

5.9 Intercepts

For the line

$$y = ax + b,$$

the intercepts are

$$(0, b)$$

on the vertical axis and

$$\left(-\frac{b}{a}, 0\right)$$

on the horizontal axis (provided $a \neq 0$).

Two points determine a line, so the intercepts alone are enough to sketch it.

5.10 Special Cases

Horizontal lines have slope zero and take the form

$$y = c.$$

Vertical lines have equation

$$x = c,$$

and cannot be written as $y = ax + b$ because their slope is undefined.

5.11 Linear Inequalities

A line cuts the plane in two: the points above it,

$$y > ax + b,$$

and the points below it,

$$y < ax + b.$$

These inequalities describe feasible regions—and feasible regions are everywhere in economic analysis.

5.12 The Budget Constraint

Suppose a consumer purchases x units of one good and y units of another. If prices are $p_x > 0$ and $p_y > 0$ and income is I, affordability requires

$$p_x x + p_y y \leq I.$$

The boundary,

$$p_x x + p_y y = I,$$

is called the budget line. Rewriting,

$$y = -\frac{p_x}{p_y}x + \frac{I}{p_y},$$

reveals that its slope equals the negative relative price,

$$-\frac{p_x}{p_y}.$$

The feasible consumption bundles lie on or below the budget line in the nonnegative quadrant.

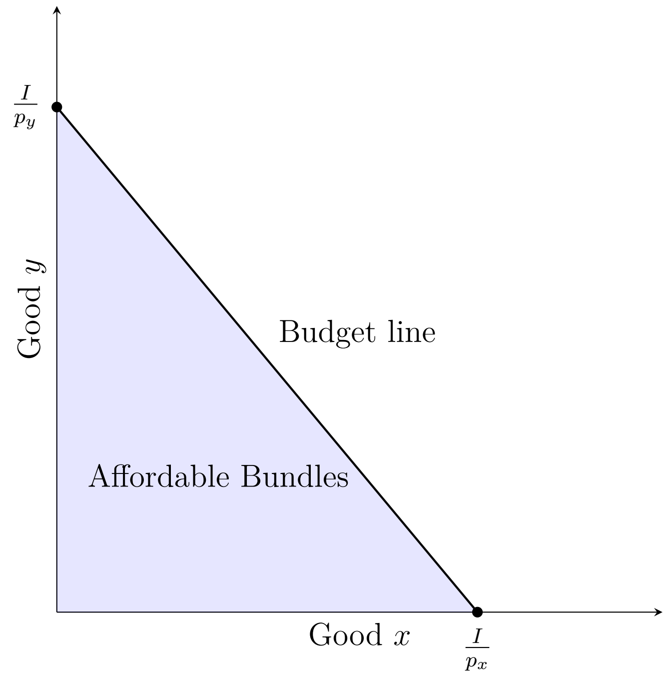


Figure 5.1: The budget constraint limits the consumer's choices. Changes in income cause parallel shifts of the line, while changes in relative prices change the slope.

Chapter Summary

By the end of this chapter, you should be able to:

- identify the domain and range of a function;
- represent a function on the Cartesian plane;
- interpret the slope and intercepts of a linear function;
- calculate a slope from two points;
- convert between direct and inverse demand functions;
- determine a line from two points or from one point and its slope;
- graph linear inequalities and identify feasible regions;
- write and interpret a consumer's budget constraint.

Linear functions provide the foundation for much of introductory economics, including demand analysis, budget constraints, comparative statics, and optimization.

Exercises

1. Find the domain and range of each function.

a. $f(x) = \sqrt{x+4}$

b. $g(x) = \frac{1}{x+2}$

c. $Q = 200 - 5P$, assuming that both price and quantity must be nonnegative

2. Evaluate the following linear functions at the specified values of x.

a. Linear function $f(x) = 2x + 1$. Evaluate $f(3)$.

b. Linear function $f(x) = -x + 4$. Evaluate $f(-2)$.

c. Linear function $f(x) = 3x - 5$. Evaluate $f(4)$.

d. Linear function $f(x) = \frac{1}{2}x + 2$. Evaluate $f(8)$.

e. Linear function $f(x) = -2x + 7$. Evaluate $f(5)$.

3. Determine whether each point lies on the graph of the given linear function.

a. Function $y = 2x + 1$. Determine whether $(3, 7)$ lies on the graph.

b. Function $y = -x + 4$. Determine whether $(2, 1)$ lies on the graph.

c. Function $y = 3x - 5$. Determine whether $(4, 8)$ lies on the graph.

d. Function $y = \frac{1}{2}x + 2$. Determine whether $(6, 5)$ lies on the graph.

e. Function $y = -2x + 7$. Determine whether $(3, 2)$ lies on the graph.

4. For each linear function below, draw the graph on the given coordinate plane.

a. $y = 2x + 1$

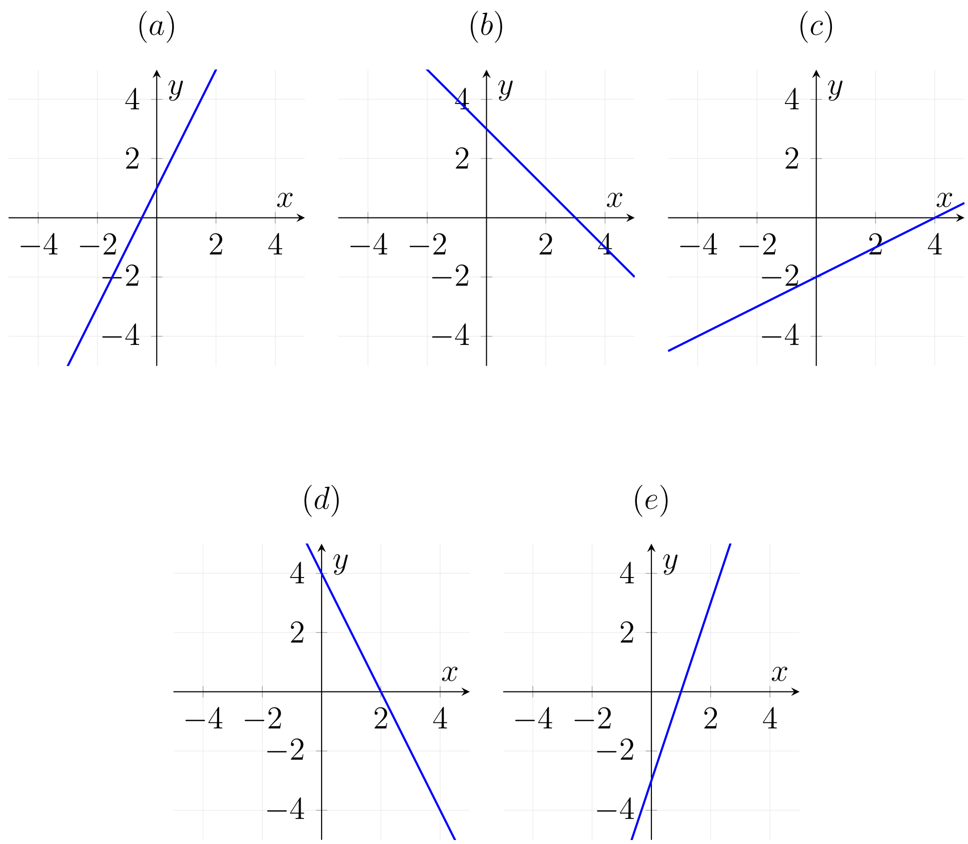
b. $y = -x + 3$

c. $y = \frac{1}{2}x - 2$

d. $y = -2x + 4$

e. $y = 3x - 3$

5. For each graph below, determine the equation of the line.



6. Calculate the slope of the line passing through the following points.

a. $A(1,3),\ B(4,9)$

b. $A(-2,5),\ B(3,0)$

c. $A(2,-1),\ B(6,7)$

d. $A(-4,-3),\ B(2,6)$

e. $A(0,4),\ B(5,2)$

7. Find the equation of the linear function that passes through the given points.

a. $A(2,5),\ B(5,11)$

b. $A(-1,4),\ B(3,-4)$

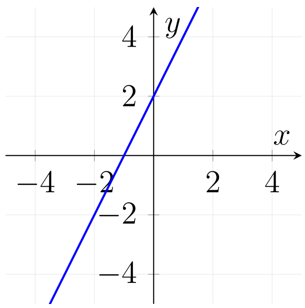
c. $A(0,-2),\ B(4,6)$

d. $A(-3,7),\ B(1,1)$

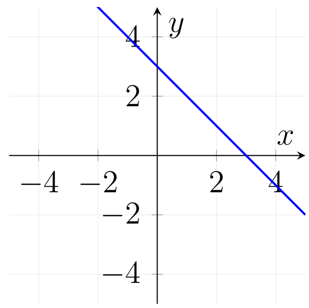
e. $A(2,-3),\ B(7,2)$

8. For each graph below, determine the x-intercept and the y-intercept.

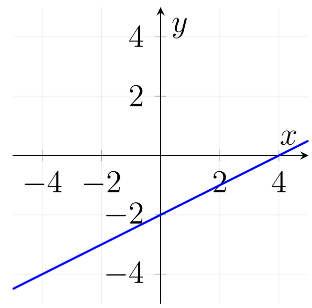
(a)



(b)



(c)



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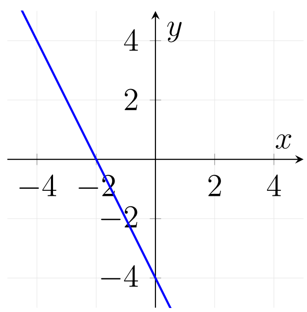
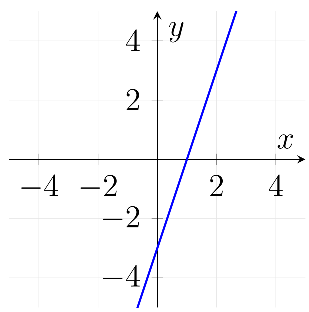


(e)



9. Consider $y = -\frac{3}{4}x + 12$.

a. What is the slope?

b. What is the y-intercept?

c. Find y when $x = 8$.

d. Find two points on the line.

e. Rewrite the equation in the form $ax + by + c = 0$.

10. A line has x-intercept 8 and y-intercept 12.

a. Write the equation in intercept form.

b. Rewrite it in the form $y = ax + b$.

c. Find the slope.

d. Does the line increase or decrease?

11. Consider $\frac{x}{5} + \frac{y}{20} = 1$.

a. Find the x-intercept.

b. Find the y-intercept.

c. Rewrite in the form $y = ax + b$.

d. Find the slope.

12. A streaming company records its annual profit (in millions of dollars) as follows: Year 2021, Profit 12; Year 2022, Profit 18; Year 2023, Profit 25; Year 2024, Profit 22; Year 2025, Profit 30. Let $P(x)$ denote the company's profit in year x.

a. What is the independent variable?

b. What is the dependent variable?

c. Find $P(2023)$.

d. Find $P(2025)$.

13. Suppose the average household consumption is given by $C(Y) = 0.75Y + 6000$ where Y = annual income (dollars) and C = annual consumption (dollars).

a. Is this a linear function?

b. Find $C(20{,}000)$.

c. Find $C(40{,}000)$.

d. How much does consumption increase when income rises from \$20,000 to \$40,000?

e. Interpret the coefficient 0.75.

14. The weekly demand for movie tickets is $Q = 500 - 8p$ where Q = tickets demanded and p = ticket price (dollars).

a. Find the quantity demanded when $p = 20$.

b. Find the quantity demanded when $p = 35$.

c. Rewrite the equation with p as a function of Q.

d. What happens to demand when price increases?

15. A firm's sales revenue (in thousands of dollars) is represented by the points $A = (10, 50)$ and $B = (25, 95)$.

a. Find the slope.

b. Interpret the slope.

16. A taxi company charges a fixed fee of \$4 plus \$2.50 per mile traveled. Let $F(m)$ denote the total fare (in dollars) for a trip of m miles.

a. Write the fare function $F(m)$.

b. Is $F(m)$ a linear function? If $F(m) = am + b$, identify a and b.

c. Find $F(12)$ and $F(20)$.

d. By how much does the fare increase if the trip length increases from 12 miles to 20 miles?

17. The annual profits (in millions of dollars) of two companies are represented by $A(x) = 5x + 10$ and $B(x) = 3x + 20$, where x is the number of years after 2020.

a. Find the profit of each company in 2024.

b. Which company has the larger slope?

c. Interpret the slope of each function.

d. In what year will the two companies have the same profit?

e. Which company has the larger profit in 2030?

18. Two products have demand functions $Q_A = 400 - 5p$ and $Q_B = 400 - 2p$.

a. Find the slope of each demand curve.

b. Which demand falls faster as price increases?

c. Find the quantity demanded when $p = 20$.

d. Which product has greater demand at $p = 20$?

19. A firm observes: (Units Sold, Revenue) = $(100, 3000)$ and $(300, 7000)$. Assume revenue is linear.

a. Find the slope.

b. Find the revenue function.

c. Find revenue when 500 units are sold.

20. A consumer has \$120. Books cost \$10 each and notebooks cost \$6 each.

a. Write the budget equation.

b. Rewrite as $y = ax + b$, where x is books and y is notebooks.

c. Find the slope.

d. Interpret the slope.

e. Find both intercepts.

Chapter 6

Systems of Linear Equations

6.1 Why Systems of Linear Equations Matter in Economics

Many economic problems have several unknowns that have to be pinned down at the same time. Prices and quantities are jointly determined in markets, firms make decisions involving multiple products, and macroeconomic models contain endogenous variables that interact. Systems of equations provide the mathematical framework for analyzing these relationships and identifying outcomes that satisfy several economic conditions at once. The equilibrium examples in this chapter illustrate this idea.

The same methods extend beyond introductory market equilibrium. Matrix algebra, input-output analysis, general equilibrium theory, econometrics, optimization, and computational economics all rely on solving systems of equations. Understanding when solutions are unique, nonexistent, or infinitely many also provides economic insight, such as whether a model determines a unique equilibrium or requires additional assumptions. This chapter prepares you for the linear algebra used in later economics courses.

6.2 Geometric Interpretation

Consider the equations

$$y = 2x + 6$$

and

$$x + y = 12.$$

Each equation is a line. The one point where they cross satisfies both at once—that's your solution.

6.3 Substitution Method

If one equation already has a variable alone on one side, substitution is usually the fastest route.

For example, with demand and supply equations

$$Q = -4P + 2100,$$

$$Q = 2P + 300,$$

equating the right-hand sides yields

$$-4P + 2100 = 2P + 300,$$

from which

$$P = 300, \qquad Q = 900.$$

6.4 Elimination Method

For equations in standard form,

$$2x + 3y = 4,$$

$$3x - 5y = -3,$$

multiply one or both equations so that one variable has opposite coefficients. Adding the equations eliminates that variable, reducing the system to a single equation.

Multiplying the first equation by 3 and the second equation by -2 gives

$$6x + 9y = 12,$$

$$-6x + 10y = 6.$$

Adding these equations eliminates x:

$$19y = 18,$$

so

$$y = \frac{18}{19}.$$

Substituting this value into the first original equation gives

$$2x + 3\left(\frac{18}{19}\right) = 4,$$

and therefore

$$x = \frac{11}{19}.$$

Thus, the solution is

$$(x, y) = \left(\frac{11}{19}, \frac{18}{19}\right).$$

Once you have one variable, plug it back into either original equation to get the other.

6.5 Economic Equilibrium

The intersection of demand and supply curves determines the equilibrium price and quantity.

In math terms, equilibrium is just the simultaneous solution of two linear equations.

Why is the intersection a resting point? Suppose the price were above P^*. Then quantity supplied would exceed quantity demanded—a surplus—and sellers, stuck with unsold goods, would cut their prices. Suppose instead the price were below P^*. Then quantity demanded would exceed quantity supplied—a shortage—and buyers competing for scarce goods would bid prices up. Only at the intersection does neither side have a reason to move. That's what makes it an equilibrium.

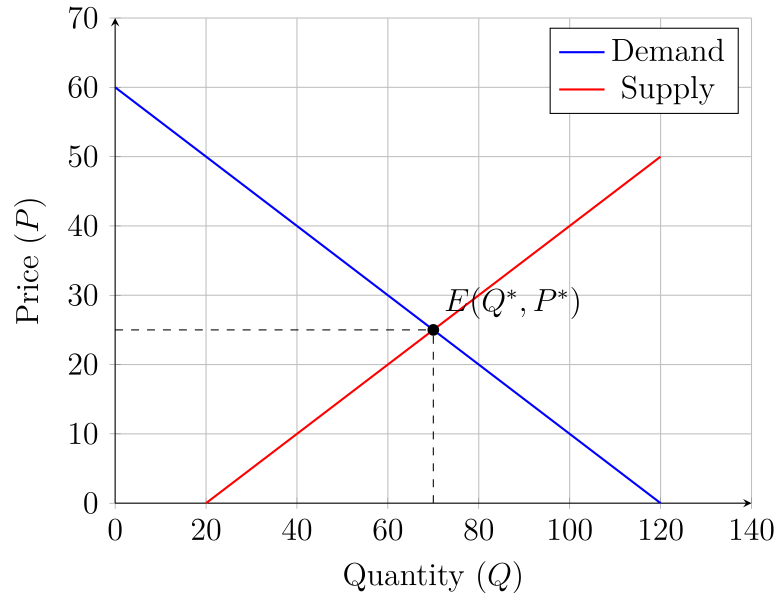


Figure 6.1: Market equilibrium occurs at the unique coordinate where the supply and demand linear functions intersect.

6.6 Existence and Uniqueness of Solutions

For two linear equations,

$$y = ax + b,$$

$$y = a'x + b',$$

exactly three things can happen.

1. If $a \neq a'$, the lines intersect once, giving a unique solution.
2. If $a = a'$ and $b = b'$, the equations describe the same line, producing infinitely many solutions.
3. If $a = a'$ but $b \neq b'$, the lines are parallel and no solution exists.

6.7 General Linear Form

Linear equations may also appear as

$$Ax + By + C = 0.$$

Provided $B \neq 0$, solving for y,

$$y = -\frac{A}{B}x - \frac{C}{B},$$

converts the equation to slope-intercept form. For example,

$$4x + 2y - 12 = 0 \iff 2y = -4x + 12 \iff y = -2x + 6.$$

Thus this line has slope -2 and vertical intercept 6; moving all terms back to the left recovers the general form. In general, the slope is

$$-\frac{A}{B}.$$

Consequently, the conditions for uniqueness, coincidence, and parallelism can also be expressed in terms of the coefficients A, B, and C.

6.8 Systems with Three Variables

A system of three unknowns generally requires three independent equations,

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1, \\ a_2x + b_2y + c_2z &= d_2, \\ a_3x + b_3y + c_3z &= d_3. \end{aligned}$$

The elimination procedure extends naturally:

1. Eliminate one variable from two pairs of equations.

2. Obtain two equations in two unknowns.

3. Solve the reduced system.
4. Substitute backward to recover the remaining variables.

6.8.1 Example

Solve the system

$$\begin{aligned} x + y + z &= 9 \quad \text{(E1)}, \\ 2x - y + z &= 5 \quad \text{(E2)}, \\ x + 2y - z &= 4 \quad \text{(E3)}. \end{aligned}$$

Step 1: Eliminate x from two pairs of equations:

$$\begin{aligned} \text{(E2)} - 2\text{(E1)}: &\quad -3y - z = -13 \quad \Longleftrightarrow \quad 3y + z = 13, \\ \text{(E3)} - \text{(E1)}: &\quad y - 2z = -5. \end{aligned}$$

Step 2: The resulting system in two unknowns is

$$3y + z = 13 \quad \text{(R1)}, \qquad y - 2z = -5 \quad \text{(R2)}.$$

Step 3: Add twice the first reduced equation to the second:

$$\begin{aligned} 2\text{(R1)} + \text{(R2)} &\quad \Longrightarrow \quad 7y = 21 \quad \Longrightarrow \quad y = 3, \\ 3(3) + z = 13 &\quad \Longrightarrow \quad z = 4. \end{aligned}$$

Step 4: Substitute backward into (E1):

$$x + 3 + 4 = 9 \quad \Longrightarrow \quad x = 2, \qquad \boxed{(x, y, z) = (2, 3, 4)}.$$

A check in the three original equations gives

$$2 + 3 + 4 = 9, \qquad 2(2) - 3 + 4 = 5, \qquad 2 + 2(3) - 4 = 4.$$

This step-by-step elimination is Gaussian elimination in miniature—the same algorithm computers use to solve big systems in applied economics.

6.9 Applications

Systems of linear equations arise throughout economics, including:

- market equilibrium,
- input–output analysis,
- national income accounting (e.g., finding equilibrium GDP in the IS-LM model),
- linear approximation of economic models,
- comparative statics with multiple endogenous variables.

Chapter Summary

By the end of this chapter, you should be able to:

- interpret the solution of a system as the intersection of equations;
- solve systems of two equations by substitution or elimination;
- determine equilibrium price and quantity from demand and supply equations;
- distinguish among unique solutions, no solutions, and infinitely many solutions;
- interpret systems written in general linear form;
- solve systems involving three variables;
- apply systems of equations to taxes, costs, and other economic problems.

Mastering systems of linear equations provides the foundation for matrix algebra, simultaneous equation models, and econometrics.

Exercises

1. Solve for x and y:

a. $y = 4x - 7$, $y = -2x + 17$

b. $y = 5x + 1$, $y = -3x + 25$

2. Solve for x and y:

a. $x + 2y = 19$, $x = 2y - 1$

b. $3x - 2y = 7$, $y = \frac{x}{2} + 4$

c. $7x + 4y = 46$, $5x - 6y = -2$

3. Solve the following systems of equations using elimination.

a. $x + y = 5$, $2x - y = 1$

b. $x + 2y = 7$, $3x - y = 4$

c. $2x + y = 8$, $x - y = 1$

d. $x + 3y = 10$, $2x + y = 8$

e. $3x + y = 11$, $x + 2y = 7$

4. Solve for x, y, and z:

a. $x + y + z = 6$, $2x - y + z = 3$, $x + 2y - z = 2$

b. $x + y + z = 7$, $2x - y + z = 9$, $3x + 2y - z = 12$

5. For each system of equations below, determine whether it has a unique solution, infinitely many solutions, or no solution. If the system has a unique solution, find it.

a. System: $x + y = 5$ and $2x - y = 1$.

b. System: $x + 2y = 6$ and $2x + 4y = 12$.

c. System: $x - y = 3$ and $2x - 2y = 10$.

d. System: $3x + y = 7$ and $6x + 2y = 14$.

e. System: $2x + 3y = 8$ and $4x + 6y = 15$.

6. Determine the number of solutions.

a. $2x + 3y = 7$, $4x + 6y = 14$

b. $3x - y = 5$, $6x - 2y = 8$

c. $y = 2x + 1$, $y = -x + 7$

7. For each market below, find the equilibrium price and equilibrium quantity by equating Q_D to Q_S.

a. $Q_D = 1200 - 40p$, $Q_S = 300 + 20p$

b. $Q_D = 1500 - 50p$, $Q_S = 200 + 30p$

c. $Q_D = 900 - 25p$, $Q_S = 100 + 15p$

d. $Q_D = 1800 - 60p$, $Q_S = 600 + 20p$

e. $Q_D = 1000 - 20p$, $Q_S = 100 + 10p$

8. For each market below, find the equilibrium price and equilibrium quantity by substituting one equation into the other.

a. $Q_D = 1000 - 20p$, $p_S = 10 + 0.05Q$

b. $Q_D = 1200 - 30p$, $p_S = 8 + 0.04Q$

c. $Q_D = 800 - 25p$, $p_S = 5 + 0.02Q$

d. $Q_D = 1500 - 50p$, $p_S = 15 + 0.03Q$

e. $Q_D = 900 - 10p$, $p_S = 12 + 0.06Q$

9. For each market below, graph the demand and supply functions on the coordinate plane. Then determine the equilibrium price and equilibrium quantity from the graph.

a. $Q_D = 100 - 2p$, $Q_S = 20 + p$

b. $Q_D = 120 - 3p$, $Q_S = 30 + p$

c. $Q_D = 90 - p$, $Q_S = 10 + 3p$

d. $Q_D = 80 - 2p$, $Q_S = 20 + 2p$

e. $Q_D = 150 - 5p$, $Q_S = 30 + p$

10. Suppose the demand function is: $Q = 2400 - 6P$ and the supply function is: $Q = 3P + 150$. Find the equilibrium price and the equilibrium quantity.

11. For each market below, find the equilibrium producer price, consumer price, and quantity when a per-unit tax creates a wedge between the price paid by consumers and the price received by producers.

a. $Q_D = 1200 - 30p_c$, $Q_S = 300 + 20p_p$, $p_c = p_p + 10$

b. $Q_D = 1500 - 50p_c$, $Q_S = 300 + 20p_p$, $p_c = p_p + 10$

c. $Q_D = 900 - 15p_c$, $Q_S = 150 + 5p_p$, $p_c = p_p + 6$

d. $Q_D = 800 - 20p_c$, $Q_S = 100 + 10p_p$, $p_c = p_p + 8$

e. $Q_D = 1000 - 20p_c$, $Q_S = 200 + 5p_p$, $p_c = p_p + 10$

12. A government imposes a per-unit tax of \$20 on producers. Before the tax: $Q_D = 2500 - 5P_c$ and $Q_S = 3P_p + 400$, where P_c = price paid by consumers and P_p = price received by producers. After the tax: $P_c = P_p + 20$.

a. Write a system of equations that describes the market equilibrium after the tax.

b. Find the equilibrium price paid by consumers.

c. Find the equilibrium price received by producers.

d. Find the equilibrium quantity.

13. For each pair of cost functions below, determine the output level at which the two costs are equal and find the common cost.

a. $C_1 = 200 + 10Q$, $C_2 = 500 + 4Q$

b. $C_1 = 300 + 8Q$, $C_2 = 700 + 2Q$

c. $C_1 = 150 + 12Q$, $C_2 = 450 + 6Q$

d. $C_1 = 100 + 15Q$, $C_2 = 400 + 5Q$

e. $C_1 = 250 + 9Q$, $C_2 = 700 + 4Q$

14. Two movie tickets and three drinks cost \$33. Four movie tickets and one drink cost \$29.

a. Define variables.

b. Find the price of each.

15. Two competing coffee shops estimate their daily sales using the following equations: Firm A: $Q_A = 600 - 4P_A + P_B$, Firm B: $Q_B = 500 - 3P_B + 2P_A$. Suppose the market is in a situation where both firms sell the same quantity and charge the same price.

a. Write a system of equations describing this situation.

b. Find the common price.

c. Find the quantity sold by each firm.

d. Find total market sales.

Chapter 7

Nonlinear and Multivariable Functions

7.1 Why Nonlinear and Multivariable Functions Matter in Economics

Linear functions are excellent first approximations, but many economic relationships are nonlinear. Diminishing marginal returns, economies of scale, price changes in general equilibrium, and the time value of money in financial economics all require nonlinear functions. Economic decisions also involve multiple variables: firms use capital and labor, consumers choose among many goods, and macroeconomic models track several indicators at once. This chapter extends the function concept to nonlinear functions of one variable and to functions of several variables. Many examples here use inverse demand functions, quadratic utility functions, Cobb-Douglas production functions, and total cost functions of several variables.

This material connects directly to intermediate economics: Cobb-Douglas production and utility functions, consumer and producer theory, and marginal analysis all require multivariable functions. The inverse-demand, aggregation, and marginal-analysis examples in this chapter lay the groundwork.

7.2 Power Functions

For any real exponent r, the function

$$y = x^r$$

is a *power function* defined for $x > 0$. If r is rational with odd denominator, the domain may extend to negative x as well.

Common shapes of $y = x^r$ for $x > 0$ include:

r	Name	Shape for $x > 0$
$r > 1$	increasing and convex	curves upward
$r = 1$	linear	straight line
$0 < r < 1$	increasing and concave	rises but flattens
$r = 0$	constant	flat line at $y = 1$
$r < 0$	decreasing and convex	falls and flattens

Two words in that table deserve precise definitions, since the exercises use them throughout. A function is *concave* on an interval if its slope is nonincreasing there—each additional unit of x adds no more than the last, so the curve bends downward like a hilltop. It is *convex* if its slope is nondecreasing—each additional unit of x adds no less than the last, so the curve bends upward like a bowl. When the slope is *strictly* decreasing (increasing), the function is *strictly* concave (convex). A quick visual test: a concave curve lies *on or above* the straight line connecting any two of its points; a convex curve lies *on or below* it.

For economists, the most important cases are $r > 1$ and $0 < r < 1$: costs that accelerate (convex) versus production and utility that display diminishing marginal returns (concave).

7.3 Exponential Functions

An exponential function is written as

$$y = b^x,$$

where the base $b > 0$ and $b \neq 1$. When $b > 1$, the function grows—that growth eventually outruns every polynomial. When $0 < b < 1$, it decays.

7.4 Logarithmic Functions

The inverse of an exponential function is a logarithmic function,

$$y = \log_b x.$$

By definition,

$$y = \log_b x \iff b^y = x.$$

For example, if $b = 10$,

$$\log_{10} 1000 = 3,$$

because $10^3 = 1000$. (For completeness, Chapter 2 covers the properties of logarithms and their applications in detail.)

7.5 Comparing Nonlinear Functions

Put the three families on one graph for $x > 0$ and the differences jump out.

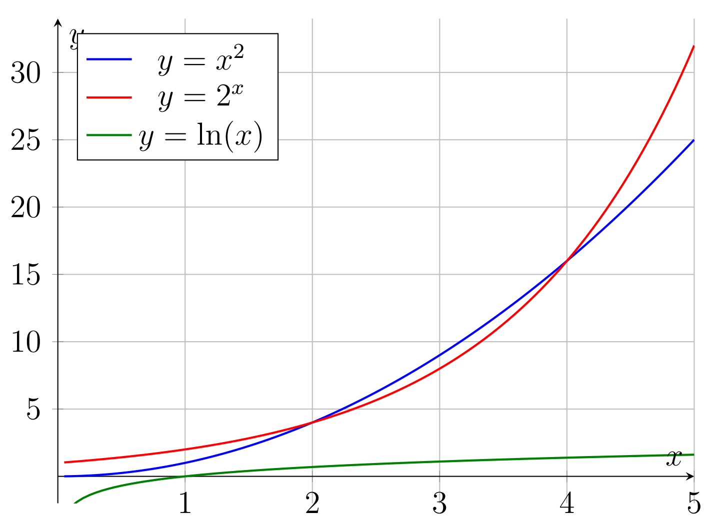


Figure 7.1: Comparing growth for $x > 0$: the exponential 2^x eventually outruns the power x^2, while $\ln(x)$ grows slowest. The logarithm is concave (its slope keeps falling); the quadratic and the exponential are convex (they keep getting steeper).

Three lessons from the picture. First, the exponential 2^x eventually outruns *every* power function—no matter how large the exponent, exponential growth wins in the end. That is why compound interest, at its core an exponential process, is so powerful. Second, the logarithm grows slowest of all: it keeps rising, but ever more sluggishly. Third, you can *see* concave and convex here. The logarithm bends downward everywhere—it is concave on $x > 0$, with a slope that keeps falling. The quadratic and the exponential bend upward—they are convex, getting steeper as x grows. Whenever a marginal quantity is diminishing—marginal utility, marginal product—you are looking at a concave function; when each extra unit costs more than the last, the function is convex.

7.6 Inverse Functions

A function f has an *inverse*—a function that undoes it—exactly when f never takes the same value twice. The inverse of f, written f^{-1}, satisfies

$$f^{-1}(f(x)) = x$$

for every x in the domain. For example, the inverse of "multiply by 3 and add 8" is "subtract 8 and divide by 3": apply one after the other and you return to where you started.

To find an inverse algebraically, write the function as $y = f(x)$ and solve for x in terms of y—the same equation-solving you practiced in Chapter 3. The result is $x = f^{-1}(y)$; renaming the variable gives $y = f^{-1}(x)$. For $y = 3x + 8$,

$$3x = y - 8, \qquad x = \frac{y-8}{3},$$

so $f^{-1}(x) = (x - 8)/3$.

Not every function has an inverse. The squaring function $y = x^2$ sends both 2 and -2 to 4, so no function can undo it on the whole real line—handed the output 4, the input is ambiguous. A function has an inverse exactly when it is *one-to-one*: distinct inputs always produce distinct outputs.

The visual test is the *horizontal line test*: a function is one-to-one if every horizontal line crosses its graph at most once. Algebraically, the cleanest guarantee is *monotonicity*. A function is *increasing* on an interval if larger inputs always give larger outputs—its graph climbs, never dipping down; it is *decreasing* if larger inputs always give smaller outputs. A function that is increasing (or decreasing) on its whole domain is automatically one-to-one, since it can never revisit a value. Note that $y = x^2$ fails the horizontal line test on all real numbers, but on the restricted domain $x \geq 0$ it is increasing and therefore invertible there.

You have already met a famous inverse pair: for $b > 1$, the exponential $y = b^x$ is increasing and hence one-to-one, and its inverse is the logarithm $y = \log_b x$ (Chapter 2).

7.6.1 Inverting Demand Functions

In economics, prices and quantities are frequently interchanged. Recall from Chapter 5 that inverting a demand function produces the inverse demand function—price as a function of quantity—and that economists conventionally graph inverse demand. You'll use this manipulation constantly, so here's the full procedure once.

Consider the demand function

$$Q = 120 - 2P.$$

Then

$$2P = 120 - Q,$$

so

$$P = 60 - \frac{Q}{2}.$$

This is the inverse demand function: price as a function of quantity.

For a nonlinear example, consider

$$Q = 50 - P^2,$$

where both Q and P are nonnegative. Then

$$P^2 = 50 - Q,$$

and taking square roots (the nonnegative root) gives

$$P = \sqrt{50 - Q}.$$

7.7 Multivariable Functions

Many economic outcomes depend on more than one variable: output depends on capital and labor, utility on the quantities of several goods, demand on price and income. A *multivariable function* takes several inputs and returns a single output, written for example as $F(K, L)$ or $D(p, m)$.

7.7.1 Evaluating Multivariable Functions

To evaluate a multivariable function, substitute each value into its corresponding variable.

For example, consider the demand function

$$D(p, m) = 20 - \frac{p}{2} + \frac{m}{10},$$

where p denotes price and m denotes income. If $p = 40$ and $m = 500$, then

$$D(40, 500) = 20 - \frac{40}{2} + \frac{500}{10} = 20 - 20 + 50 = 50.$$

7.7.2 The Cobb-Douglas Production Function

One of the most widely used functions in economics is the Cobb-Douglas production function,

$$F(K, L) = AK^a L^b,$$

where K and L denote capital and labor, $A > 0$ denotes technology, and a and b are positive parameters. The marginal product of each input—how much output rises when that input rises while the other is held fixed—comes from applying the power rule one variable at a time, exactly as the marginal-analysis preview at the end of this chapter shows.

7.7.3 Returns to Scale

If a firm doubles all its inputs, does output double? For a production function $F(K, L)$ and a scale factor $t > 1$, compare $F(tK, tL)$ with $tF(K, L)$:

- *constant returns to scale* if $F(tK, tL) = tF(K, L)$—doubling inputs doubles output;
- *increasing returns to scale* if $F(tK, tL) > tF(K, L)$;
- *decreasing returns to scale* if $F(tK, tL) < tF(K, L)$.

For the Cobb-Douglas function the test is immediate:

$$F(tK, tL) = A(tK)^a(tL)^b = t^{a+b}AK^aL^b = t^{a+b}F(K, L),$$

so everything hinges on the exponent sum $a + b$:

- $a + b = 1$: constant returns to scale;
- $a + b > 1$: increasing returns to scale;
- $a + b < 1$: decreasing returns to scale.

For example, $F(K, L) = 5K^{0.4}L^{0.6}$ has $a + b = 1$: constant returns—doubling K and L exactly doubles output. $F(K, L) = 2K^{0.7}L^{0.5}$ has $a + b = 1.2 > 1$: increasing returns. $F(K, L) = 4K^{0.3}L^{0.5}$ has $a + b = 0.8 < 1$: decreasing returns.

7.8 Aggregating Individual Demands and Supplies

7.8.1 Horizontally Aggregating Individual Demands

To find market demand, add quantities at each price—that is, sum horizontally, across individuals. This is the horizontal summation of individual demand curves.

Consider, for example, two consumers:

$$Q_1 = 100 - 2P,$$

$$Q_2 = 50 - P.$$

The market demand is

$$Q_M = Q_1 + Q_2 = (100 + 50) - (2P + P) = 150 - 3P.$$

The inverse market demand is

$$P = 50 - \frac{Q_M}{3}.$$

This horizontal summation underlies consumer surplus calculations: aggregating individual demands into market demand is the first step in measuring total consumer surplus.

When individual demand curves hit zero at different prices, the market demand curve has a kink: one consumer exits before the other. Exercises 11–13 explore this idea and its inverse-demand counterpart, with full worked solutions in the back.

7.8.2 Horizontally Aggregating Individual Supplies

Supply aggregates exactly the way demand does: at each price, add the quantities every firm supplies—sum horizontally.

For example, suppose two firms have supply functions

$$Q_1 = 10 + 2P, \qquad Q_2 = 5 + P.$$

Market supply is

$$Q_S = Q_1 + Q_2 = (10 + 5) + (2P + P) = 15 + 3P,$$

with inverse supply $P = (Q_S - 15)/3$. As a check, at $P = 10$ the firms supply 30 and 15, for a total of 45—and $15 + 3(10) = 45$.

7.8.3 Vertical Aggregation: Public Goods

Some goods are consumed jointly—a park, national defense, clean air. Everyone consumes the *same* quantity, so you aggregate the other way: at each quantity, add up what each person would pay—sum vertically.

Suppose two neighbors value a public park, with inverse demands $P = 50 - 2Q$ and $P = 30 - Q$, where Q is the park's size in acres. Vertically summing willingness to pay,

$$P_M = (50 - 2Q) + (30 - Q) = 80 - 3Q.$$

This holds while both neighbors have positive willingness to pay—neighbor A's reaches zero at $Q = 25$, neighbor B's at $Q = 30$, so the sum applies for $0 \leq Q \leq 25$, beyond which only B's demand counts. The economic interpretation is the point: because everyone enjoys the same quantity of a public good, the social value of one more unit is the sum of what each person would pay for it—which is exactly what vertical aggregation computes.

7.9 A Preview: Marginal Analysis

In economics, a *marginal* quantity answers one question: how much does the outcome change when a single input changes and everything else stays fixed? Answering that question properly takes calculus, which is the subject of Volume II. But the core idea is simple enough to preview now—and it shows where the functions in this chapter are heading.

The idea: freeze the other variables—treat them as constants—and apply one new rule to each term, one variable at a time. The rule, called the *power rule for derivatives*, says that x^n becomes $n\,x^{n-1}$: multiply by the old exponent, then reduce the exponent by one. (Chapter 2 taught you rules for *rewriting* powers, such as $x^a x^b = x^{a+b}$; this is a different rule, about *rates of change*. You will see why it works in Volume II.)

Consider the quadratic utility function

$$U(X, Y) = 12X + 18Y - X^2 - \frac{3}{2}Y^2.$$

The marginal utility of X holds Y fixed. Term by term: $12X$ gives 12, $-X^2$ gives $-2X$, and every pure-Y term behaves as a constant and drops out. Hence

$$MU_X = 12 - 2X, \qquad MU_Y = 18 - 3Y.$$

At $X = 4$, $Y = 6$: $U(4, 6) = 48 + 108 - 16 - 54 = 86$, $MU_X(4, 6) = 12 - 8 = 4$, and $MU_Y(4, 6) = 18 - 18 = 0$—at $Y = 6$, an additional infinitesimal increase in Y adds nothing at the margin. (The discrete sixth unit still added something: $U(4, 6) - U(4, 5) = 86 - 84.5 = 1.5$. Marginal utility measures the rate of change *at* a point, not the jump *between* points.)

For production, take $f(K, L) = 2K^2 + 3KL + 4L^2$. Holding L fixed, the marginal product of capital is

$$MP_K = 4K + 3L,$$

and holding K fixed,

$$MP_L = 3K + 8L.$$

At $K = 10$, $L = 20$: $f(10, 20) = 200 + 600 + 1600 = 2400$, $MP_K(10, 20) = 40 + 60 = 100$, and $MP_L(10, 20) = 30 + 160 = 190$.

For costs, $C(x, y) = 5x^2 + 3xy + 2y^2$ gives marginal costs

$$MC_x = 10x + 3y, \qquad MC_y = 3x + 4y.$$

At $x = 6$, $y = 8$: $C(6, 8) = 180+144+128 = 452$, $MC_x(6, 8) = 60+24 = 84$, and $MC_y(6, 8) = 18 + 32 = 50$. A multi-product firm with $C(Q_1, Q_2) = 100 + 3Q_1 + 4Q_2 + \frac{1}{2}Q_1^2 + \frac{1}{2}Q_2^2$ has $MC_1 = 3 + Q_1$ and $MC_2 = 4 + Q_2$: the fixed cost 100 and the other product's terms act as constants.

This is only a preview. The power rule you just applied, the ∂ notation, and the sum, product, and chain rules—with the proofs of why they work—belong to Volume II, the *Intermediate Toolkit*. What you just did is exactly what those rules formalize: vary one variable, freeze the rest.

Chapter Summary

By the end of this chapter, you should be able to:

- recognize power functions and their graphs, and apply the definitions of concave and convex;
- evaluate and interpret exponential and logarithmic functions, and compare their growth rates;
- find inverse functions, and use monotonicity and the horizontal line test to decide when inverses exist;
- invert demand functions;
- evaluate multivariable functions;
- work with Cobb-Douglas production functions and classify returns to scale;
- aggregate individual demands and supplies horizontally, and aggregate vertically for public goods;
- compute marginal quantities—marginal utility, marginal product, marginal cost—as a preview of the calculus in Volume II.

Nonlinear and multivariable functions provide the mathematical language for intermediate microeconomics and macroeconomics.

Exercises

1. Evaluate each power function.

a. $8^{2/3}$

b. $27^{1/3}$

c. $4^{-1/2}$

d. $(-8)^{2/3}$

e. $16^{3/4}$

2. For each power function below, determine whether it increases or decreases for $x > 0$, and whether it is concave or convex.

a. $y = x^{0.5}$

b. $y = x^{2.5}$

c. $y = x^{-1.5}$

d. $y = x^{1.0}$

e. $y = x^{0.0}$

3. For the Cobb-Douglas production function $F(K, L) = 5K^{0.4}L^{0.6}$, evaluate at:

a. $K = 100$, $L = 100$

b. $K = 200$, $L = 150$

4. Find the inverse of each function (solve for x as a function of y):

a. $y = 3x + 8$

b. $y = x^3 - 2$

c. $y = \frac{x+4}{2}$

d. $y = e^x$, for $y > 0$

5. For each function, state whether it is increasing or decreasing on the given domain, and whether it is one-to-one:

a. $y = x^2$, all real x

b. $y = x^2$, $x \geq 0$

c. $y = x^3$, all real x

d. $y = 2^{-x}$, all real x

e. $y = \ln x$, $x > 0$

6. Find the inverse demand function (solve for P as a function of Q):

a. $Q = 200 - 4P$

b. $Q = 150 - 3P$

c. $Q = 300 - 2P$

d. $Q = 180 - 6P$

e. $Q = 250 - 5P$

7. Find the inverse demand function for each nonlinear demand:

a. $Q = 100 - P^2$, assuming $Q, P \geq 0$

b. $Q = 64 - \sqrt{P}$, assuming $Q, P \geq 0$

c. $Q = 50 - 2\sqrt{P}$, assuming $Q, P \geq 0$

8. Given the market demand function $Q_D = 200 - 4P$, find:

a. the inverse demand function

b. the maximum price

c. the maximum quantity

9. For each Cobb-Douglas production function, determine whether it exhibits constant, increasing, or decreasing returns to scale:

a. $F(K, L) = 3K^{0.4}L^{0.6}$

b. $F(K, L) = 2K^{0.7}L^{0.5}$

c. $F(K, L) = 4K^{0.3}L^{0.5}$

d. $F(K, L) = K^{0.5}L^{0.5}$

10. Two neighbors share a public park. Neighbor A's inverse demand for park acres is $P = 50 - 2Q$ and neighbor B's is $P = 30 - Q$.

a. Write the vertically aggregated inverse demand $P_M(Q)$ for the range where both neighbors have positive willingness to pay.

b. At what Q does each neighbor's willingness to pay reach zero?

c. What is the aggregate willingness to pay for 10 acres?

11. (Challenge) Two consumers have demands $Q_1 = 100-2p$ and $Q_2 = 50-p$ where p is the market price in dollars. Find the market demand function and its inverse. *Hint: check whether the two consumers exit the market at the same price or at different prices.*

12. (Challenge) Two consumers have demands $Q_1 = 50 - 2P$ and $Q_2 = 30 - P$. Find the market demand and the inverse market demand, taking into account that the demand curves have different intercepts. *Hint: start by finding the price at which each consumer's demand reaches zero.*

13. (Challenge) Two consumers have demands $Q_1 = 30 - 2p$ and $Q_2 = 40 - p$. Find the market demand function and the inverse market demand function.

14. For the utility function $U(X, Y) = 12X + 18Y - X^2 - \frac{3}{2}Y^2$, find:

a. the marginal utility of X

b. the marginal utility of Y

c. $U(4,6)$

d. $MU_X(4,6)$ and $MU_Y(4,6)$

15. For the production function $f(K,L) = 2K^2 + 3KL + 4L^2$, find:

a. the marginal product of K

b. the marginal product of L

c. $f(10,20)$

d. $MP_K(10,20)$ and $MP_L(10,20)$

16. For the cost function $C(x,y) = 5x^2 + 3xy + 2y^2$, find:

a. the marginal cost of x

b. the marginal cost of y

c. $C(6,8)$

d. $MC_x(6,8)$ and $MC_y(6,8)$

Chapter 8

Financial Mathematics and Summation

8.1 Why Financial Mathematics Matters in Economics

This chapter develops the financial mathematics and summation techniques economists use constantly: compound interest, present value, annuities, perpetuities, and the summation machinery that ties them together. These tools appear across macroeconomics, finance, and public policy.

In macroeconomics, compound growth drives long-run output dynamics: small differences in growth rates compound into enormous differences in living standards. Investment appraisal in firms and governments relies on present value and net present value. Bond pricing is discounted-cash-flow analysis. And in intertemporal economics, discounting captures how households and firms value future consumption relative to present consumption.

8.2 Interest and Compound Interest

8.2.1 Simple Interest

Simple interest is computed only on the principal P. Interest is Prt, so the future value after t years is

$$FV = P + Prt = P(1 + rt).$$

For example, borrowing \$2000 for 3 years at a simple interest rate of 4% per year yields

$$FV = 2000\Big[1 + 0.04(3)\Big] = 2000(1.12) = 2240.$$

That is, you repay \$2240: the \$2000 principal plus \$240 of interest.

8.2.2 Compound Interest

Compound interest is computed on the principal plus accumulated interest—interest earns interest. This is the central idea of this chapter, and it dominates finance and macroeconomics.

If \$1000 is invested at 5% per year, compounded annually, then after 1 year the balance is $1000(1.05) = 1050$. After 2 years it is $1050(1.05) = 1000(1.05)^2 = 1102.50$. In general, after t years the future value is

$$FV = P(1 + r)^t.$$

For example, \$1000 invested at 5% per year for 10 years grows to

$$FV = 1000(1.05)^{10} = 1000(1.6288946) \approx 1628.89.$$

With simple interest, the same investment would yield only $1000[1 + 0.05(10)] = 1500$. The extra \$128.89 is interest earning interest—compounding.

The Rule of 72 (a handy aside). A quick way to estimate how long money takes to double: divide 72 by the annual percentage rate. At 6%, money doubles in roughly $72/6 = 12$ years; at 8%, in about 9 years. (The exact doubling time is $\ln 2/\ln(1 + r)$, which is very close to $72/(100r)$ for rates between 5% and 10%.)

8.2.3 Continuous Compounding

Compounding need not happen once a year: it can happen monthly, daily, or—in the limit—continuously. If interest is compounded m times per year at nominal rate r, the future value after t years is $P(1+r/m)^{mt}$. Letting $m \to \infty$ gives continuous compounding:

$$FV = Pe^{rt}.$$

For example, \$1000 invested at 5% for 10 years with continuous compounding grows to

$$FV = 1000e^{0.05\cdot 10} = 1000e^{0.5} \approx 1000(1.6487213) \approx 1648.72.$$

With annual compounding the same investment reached \$1628.89, so compounding more often adds about \$19.83 here—the limit of "interest on interest" arriving ever faster.

8.3 Present Value

Present value answers the question: how much is a future payment worth today? The answer is the amount that would grow to that payment at the prevailing interest rate. If the interest rate is r per period and a payment C arrives in t periods, its present value is

$$PV = \frac{C}{(1+r)^t}.$$

For example, the present value of \$1000 received in 5 years at an interest rate of 4% is

$$PV = \frac{1000}{(1.04)^5} = \frac{1000}{1.2166529} \approx 821.93.$$

The interest rate r is also called the *discount rate*.

8.3.1 Variable Interest Rates

If interest rates change from year to year, each year's discount factor multiplies separately. With rates $r_1, r_2, \ldots, r_n$ and payments $\pi_1, \pi_2, \ldots, \pi_n$, the present value is

$$PV = \sum_{t=1}^{n} \frac{\pi_t}{(1+r_1)(1+r_2)\cdots(1+r_t)}.$$

For example, with rates $r_1 = 5\%$, $r_2 = 6\%$, and $r_3 = 4\%$ and payments 100, 200, and 300:

$$PV = \frac{100}{1.05} + \frac{200}{(1.05)(1.06)} + \frac{300}{(1.05)(1.06)(1.04)} = \frac{100}{1.05} + \frac{200}{1.113} + \frac{300}{1.15752}.$$

This yields

$$PV \approx 95.24 + 179.69 + 259.18 = 534.11.$$

8.3.2 Net Present Value

Present value lets you judge investments. Suppose a project costs C_0 today and generates cash flows $C_1, \ldots, C_n$ in future years. Its *net present value* is the present value of everything it pays minus what it costs:

$$NPV = -C_0 + \sum_{t=1}^{n} \frac{C_t}{(1+r)^t}.$$

The decision rule is simple: undertake the project when $NPV > 0$, reject it when $NPV < 0$, and be indifferent on value grounds when $NPV = 0$. A positive NPV means the project earns more than the discount rate requires—it creates value.

For example, a machine costs \$10,000 today and generates \$4,000 at the end of each of the next 3 years. At a 6% discount rate,

$$NPV = -10000 + \frac{4000}{1.06} + \frac{4000}{(1.06)^2} + \frac{4000}{(1.06)^3} = -10000 + 3773.58 + 3559.99 + 3358.48 \approx 692.05.$$

Since $NPV > 0$, the machine is worth buying: it creates about \$692 of value.

8.4 Geometric Series

Many financial formulas are geometric series: each term is the previous term times a common ratio. The geometric series

$$a + ar + ar^2 + \cdots + ar^{n-1}$$

has sum

$$a\,\frac{1-r^n}{1-r}.$$

This is the engine behind the annuity formulas in the next section: each one is a geometric series in disguise.

For example,

$$2 + 4 + 8 + 16 + 32 = 2\,\frac{1-2^5}{1-2} = 62.$$

8.5 Annuities

An annuity is a sequence of equal payments at regular intervals.

8.5.1 Future Value of an Ordinary Annuity

If payments of size C are made at the end of each period for n periods and the interest rate is r, the future value is

$$FV = C\,\frac{(1+r)^n - 1}{r}.$$

This is the geometric-series sum with first term $a = C$ and ratio $1+r$: $C + C(1+r) + \cdots + C(1+r)^{n-1} = C\,\frac{1-(1+r)^n}{1-(1+r)}$.

For example, \$500 deposited at the end of each year for 10 years at 6% grows to

$$FV = 500\,\frac{(1.06)^{10} - 1}{0.06} = 500(13.1807949) \approx 6590.40.$$

8.5.2 Present Value of an Ordinary Annuity

The present value of the same annuity is

$$PV = C\,\frac{1 - (1+r)^{-n}}{r}.$$

Again a geometric series, now with first term $a = C/(1+r)$ and ratio $1/(1+r)$: discounting each payment gives $\frac{C}{1+r} + \frac{C}{(1+r)^2} + \cdots + \frac{C}{(1+r)^n}$.

For example, the present value of $500 received at the end of each year for 10 years at 6% is

$$PV = 500\,\frac{1 - (1.06)^{-10}}{0.06} = 500(7.3600871) \approx 3680.04.$$

8.5.3 Annuity Due

In an annuity due, payments occur at the beginning of each period rather than the end.

For example, the future value of an annuity due of $500 per year for 10 years at 6% is

$$FV = 500\,\frac{(1.06)^{10} - 1}{0.06}\,(1.06) = 500(13.1807949)(1.06) \approx 6985.82.$$

Notice this is just the ordinary-annuity future value multiplied by (1.06): every payment gets one extra year of compounding.

8.6 Perpetuities

A perpetuity pays a fixed amount forever. If C is paid at the end of each period and the interest rate is r,

$$PV = \frac{C}{r}.$$

Equivalently, this is the infinite geometric series $\frac{C}{1+r} + \frac{C}{(1+r)^2} + \cdots$ with ratio $1/(1+r) < 1$: its sum is $\frac{a}{1-\text{ratio}} = \frac{C/(1+r)}{r/(1+r)} = \frac{C}{r}$.

For example, a perpetuity paying \$100 per year at a 5% interest rate has

$$PV = \frac{100}{0.05} = 2000.$$

The intuition: with \$2000 invested at 5%, the annual interest is exactly \$100—you can collect the \$100 each year forever without ever touching the principal.

8.6.1 Perpetuity Due

In a perpetuity due, payments begin immediately (at time 0) rather than at the end of the first period.

The intuition is worth spelling out: a perpetuity due is just one payment today, followed by an ordinary perpetuity. Today's payment isn't discounted at all, so

$$PV_{\text{due}} = C + \frac{C}{r}.$$

For example, a perpetuity due paying \$100 per year at a 5% interest rate has

$$PV = \frac{100}{0.05} + 100 = 2000 + 100 = 2100.$$

8.7 Summation Notation

Summation notation provides a compact way to express sums. The Greek letter Σ (capital sigma) denotes summation:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

For example,

$$\sum_{i=1}^{4} i^2 = 1 + 4 + 9 + 16 = 30,$$

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n.$$

8.7.1 Properties of Summation

Several rules simplify working with summations:

1. $\sum_{i=1}^{n} c = nc$, where c is a constant.
2. $\sum_{i=1}^{n} ca_i = c \sum_{i=1}^{n} a_i$.
3. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$.

For example,

$$\sum_{i=1}^{50} 3 = 150,$$

$$\sum_{i=1}^{n} 2x_i = 2 \sum_{i=1}^{n} x_i.$$

Chapter Summary

By the end of this chapter, you should be able to:

- compute future values under simple, compound, and continuous compounding;
- discount future payments to present value;
- evaluate investment projects using net present value;
- handle varying interest rates;
- value annuities and annuities due;

- value perpetuities and perpetuities due;
- use summation notation and its properties;
- sum geometric series.

Financial mathematics provides essential tools for macroeconomics, finance, and intertemporal decision-making.

Exercises

1. Compute the future value of an investment.

a. \$1000 at 5% simple interest for 3 years

b. \$2000 at 4% simple interest for 5 years

c. \$1500 at 6% simple interest for 2 years

2. Compute the future value of an investment with compound interest.

a. \$1000 at 5% per year for 3 years

b. \$2000 at 4% per year for 5 years

c. \$1500 at 6% per year for 2 years

d. \$3000 at 7% per year for 10 years

e. \$5000 at 3% per year for 8 years

3. Compute the present value of a future payment.

a. \$1000 in 3 years at 5%

b. \$2000 in 5 years at 4%

c. \$1500 in 2 years at 6%

d. \$5000 in 10 years at 7%

e. \$3000 in 8 years at 3%

4. Use the present value formula $PV = \sum_{t=1}^{n} \frac{\pi_t}{(1+r_1)(1+r_2)\cdots(1+r_t)}$ to discount future profits when interest rates vary over time.

a. A project generates a profit of \$1,000 in Year 1 and \$1,200 in Year 2. The interest rate is 3% in Year 1 and 4% in Year 2. What is the present value of the project?

b. A project generates a profit of \$800 in Year 1, \$900 in Year 2, and \$1,000 in Year 3. The interest rates are 2%, 3%, and 4%, respectively. What is the present value of the project?

c. A project generates a profit of \$1,500 in Year 1 and \$2,000 in Year 2. The interest rates are 5% in Year 1 and 6% in Year 2. What is the present value of the project?

d. A project generates a profit of \$2,000 in Year 1, \$2,000 in Year 2, and \$2,000 in Year 3. The interest rates are 4%, 4.5%, and 5%, respectively. What is the present value of the project?

e. A project generates a profit of \$1,000 in Year 1, \$1,500 in Year 2, \$2,000 in Year 3, and \$2,500 in Year 4. The interest rates are 3%, 3.5%, 4%, and 4.5%, respectively. What is the present value of the project?

5. Compute the future value of an ordinary annuity.

a. \$100 per year for 5 years at 5%

b. \$200 per year for 10 years at 4%

c. \$300 per year for 8 years at 6%

d. \$500 per year for 12 years at 7%

6. Compute the present value of an ordinary annuity.

a. \$100 per year for 5 years at 5%

b. \$200 per year for 10 years at 4%

c. \$300 per year for 8 years at 6%

d. \$500 per year for 12 years at 7%

7. Compute the future value of an annuity due.

a. \$100 per year for 5 years at 5%

b. \$200 per year for 10 years at 4%

8. Compute the present value of a perpetuity.

a. \$100 per year at 5%

b. \$200 per year at 4%

c. \$50 per year at 2.5%

9. Compute the present value of a perpetuity due.

a. \$100 per year at 5%

b. \$200 per year at 4%

c. \$50 per year at 2.5%

10. Evaluate each summation.

a. $\sum_{i=1}^{5} i$

b. $\sum_{i=1}^{4}(2i+1)$

c. $\sum_{i=1}^{3} i^2$

d. $\sum_{i=1}^{10} 5$

e. $\sum_{i=1}^{6} 3i$

11. Use summation properties to simplify and evaluate.

a. $\sum_{i=1}^{100} 4$

b. $\sum_{i=1}^{20}(3i+2)$

c. $\sum_{i=1}^{10}(i^2+3i)$

12. Sum the geometric series.

a. $1+2+4+8+16$

b. $3+6+12+24$

c. $100+50+25+12.5$

d. $5+5(1.05)+5(1.05)^2+5(1.05)^3$

13. Compute the future value with continuous compounding.

a. \$1000 at 5% for 10 years

b. \$2000 at 4% for 5 years

c. \$5000 at 6% for 3 years

14. A project costs \$5000 today and pays \$2000 at the end of each of the next 3 years. Compute its net present value and decide whether the project should be undertaken.

a. Discount rate 5%

b. Discount rate 10%

Chapter 9

Solutions to Exercises

Chapter 1: Arithmetic Foundations for Economics

1. Every integer is rational (write it over 1) and real; the naturals are the nonnegative integers:

a. -6: an integer, hence also rational and real.

b. 0: a natural number (taking $\mathbb{N} = \{0, 1, 2, \dots\}$), hence an integer, rational, and real.

c. $\frac{9}{14}$: a ratio of integers, so rational and real (not an integer).

d. $2.75 = \frac{11}{4}$: rational and real.

e. $\sqrt{5}$: not expressible as a ratio of integers, so irrational and real.

f. $-0.414141\dots$: a repeating decimal equals a fraction, so rational and real.

2.

a. 12: positive integer, so $\mathbb{N}$, $\mathbb{Z}$, rational, $\mathbb{R}$.

b. π: irrational, so $\mathbb{R}$ only (among these categories).

c. $\frac{3}{4}$: rational and $\mathbb{R}$.

d. -7: an integer but not a natural number, so $\mathbb{Z}$, rational, $\mathbb{R}$.

e. $\sqrt{5}$: irrational, so $\mathbb{R}$ only.

3. Cancel or collect first, then simplify:

a. $\frac{10x}{5} = 2x$.

b. $4x \times 3y = (4 \cdot 3)(xy) = 12xy$.

c. $2x \times x \times 5 = (2 \cdot 5)(x \cdot x) = 10x^2$.

4. Parentheses first, then multiplication/division left to right, then addition/subtraction:

a. $6^2 + 4 \times (10 - 6) = 36 + 4 \times 4 = 36 + 16 = 52$.

b. $\frac{3^4}{6} \times 3 - 2^2 = \frac{81}{6} \times 3 - 4 = 40.5 - 4 = 36.5 = \frac{73}{2}$.

c. $\frac{3^4}{6 \times 3} - 2^2 = \frac{81}{18} - 4 = 4.5 - 4 = \frac{1}{2}$.

d. $(4 + (10 - 2^2) \times 3) + 59 - 2 \times 4 = (4 + (10 - 4) \times 3) + 59 - 8 = (4 + 18) + 51 = 73$.

5. Simplify each exponent and root before combining:

a. $3 \times 5 - 3 \times 3 - 3 = 15 - 9 - 3 = 3$.

b. $(2 + \sqrt{7 - 3})^2 = (2 + \sqrt{4})^2 = (2 + 2)^2 = 16$.

c. $\frac{16 + 3 \times (4 - 4^2)}{25 - (7 \times 3)} = \frac{16 + 3(4 - 16)}{25 - 21} = \frac{16 - 36}{4} = \frac{-20}{4} = -5$.

d. Collect the exponents first:

$$2^{3+1} + 4^{4-1} \times 5^{6-2\times3} + \left(\frac{729}{9^2 \times 3} + 1\right) = 2^4 + 4^3 \times 5^0 + \left(\frac{729}{243} + 1\right) = 16 + 64 + (3 + 1) = 84.$$

e. Simplify the exponents top and bottom, then divide:

$$\text{Numerator: } 8^{\frac{42}{7} - (2+3)} \times \frac{(-9)^2}{3} - 1 \times 4^{2\times2-2} = 8^{6-5} \times \frac{81}{3} - 4^2 = 8 \times 27 - 16 = 200,$$
$$\text{Denominator: } 7^{\frac{21}{7}} + ((6 + 5) \times 5 + 2) = 7^3 + (55 + 2) = 343 + 57 = 400,$$

so the fraction is $\frac{200}{400} = \frac{1}{2}$.

6.

a. Multiplication before addition: $3 + 4 \times 2 = 3 + 8 = 11$.

b. Common denominator 4: $\frac{3}{4} + \frac{1}{2} = \frac{3}{4} + \frac{2}{4} = \frac{5}{4}$.

c. $2.5(4) - 1.2 = 10 - 1.2 = 8.8$.

d. Do the denominator first: $\frac{18}{3+3} = \frac{18}{6} = 3$.

e. $1.5 + \frac{3}{5} = 1.5 + 0.6 = 2.1$.

7. Find a common denominator for each pair:

a. $\frac{3}{8} + \frac{5}{12} = \frac{9}{24} + \frac{10}{24} = \frac{19}{24}$.

b. $\frac{7}{9} - \frac{2}{15} = \frac{35}{45} - \frac{6}{45} = \frac{29}{45}$.

c. $\frac{11}{18} + \frac{5}{24} = \frac{44}{72} + \frac{15}{72} = \frac{59}{72}$.

d. $\frac{13}{20} - \frac{3}{8} = \frac{26}{40} - \frac{15}{40} = \frac{11}{40}$.

e. $\frac{4}{7} + \frac{5}{14} - \frac{3}{21} = \frac{24}{42} + \frac{15}{42} - \frac{6}{42} = \frac{33}{42} = \frac{11}{14}$.

8. To divide, multiply by the reciprocal—and cancel before multiplying:

a. $\frac{7}{15} \div \frac{14}{25} = \frac{7}{15} \times \frac{25}{14} = \frac{7\times25}{15\times14} = \frac{175}{210} = \frac{5}{6}$.

b. $\frac{9}{16} \div \frac{27}{40} = \frac{9}{16} \times \frac{40}{27} = \frac{360}{432} = \frac{5}{6}$.

c. $\frac{5}{18} \times \frac{36}{25} = \frac{180}{450} = \frac{2}{5}$ (cancel 18 with 36 and 5 with 25 first if you like).

9.

a. $\frac{3}{8} = 3 \div 8 = 0.375$.

b. $0.75 = \frac{75}{100} = \frac{3}{4}$ in lowest terms.

c. $\frac{7}{20} = 7 \div 20 = 0.35$.

d. $1.25 = \frac{125}{100} = \frac{5}{4}$ in lowest terms.

e. $\frac{9}{4} = 9 \div 4 = 2.25$.

10. Line up the decimal points and add column by column:

a. $0.47 + 0.083 = 0.553$.

b. $3.6 + 0.045 + 0.209 = 3.854$.

c. $12.08 + 0.7 + 0.056 = 12.836$.

d. $0.009 + 0.031 + 0.6 = 0.64$.

11. Multiply as whole numbers, then place the decimal point so the answer has as many decimal places as the two factors combined:

a. 2.4×3.5: $24 \times 35 = 840$, two decimal places, so $8.40 = 8.4$.

b. 0.8×6.2: $8 \times 62 = 496$, two decimal places, so 4.96.

c. 1.25×4.8: $125 \times 48 = 6000$, three decimal places, so $6.000 = 6$.

d. 0.35×0.4: $35 \times 4 = 140$, three decimal places, so $0.140 = 0.14$.

e. 12.5×0.24: $125 \times 24 = 3000$, three decimal places, so $3.000 = 3$.

12. Look at the first digit you are dropping: 5 or more rounds the keeper digit up.

a. 3.768: the hundredths digit is $6 \geq 5$, so 3.8.

b. 12.43: the tenths digit is $4 < 5$, so 12.

c. 0.857: the thousandths digit is $7 \geq 5$, so 0.86.

d. 45.149: the hundredths digit is $4 < 5$, so 45.1.

e. 8.995: the thousandths digit is 5, so round up: 9.00.

13.

a. Exact: $\$275 \times 1.18 = \$324.50$.

b. Rounded rate: $\$275 \times 1.20 = \$330$.

c. The early rounding overstates the price by $\$330 - \$324.50 = \$5.50$. This is the whole point: rounding the *rate* before multiplying magnifies the error by the full base amount.

14.

a. $\frac{\text{consumption}}{\text{budget}} = \frac{750}{1000} = \frac{3}{4}$.

b. $\frac{480 \text{ units}}{24 \text{ labor-hours}} = 20$ units per labor-hour.

c. One notebook costs $\$12.50 \div 5 = \$2.50$, so eight cost $8 \times \$2.50 = \$20$. (Equivalently, $\frac{12.50}{5} = \frac{x}{8}$ gives $x = 20$.)

d. $\$0.80/\text{lb} \times 2{,}000\ \text{lb/ton} = \$1{,}600$ per ton.

15. Percentage change is $\frac{\text{new}-\text{old}}{\text{old}} \times 100\%$; don't confuse it with percentage *points*:

a. $\frac{540-500}{500} \times 100\% = 8\%$.

b. $\frac{225-250}{250} \times 100\% = -10\%$.

c. $5\% - 4\% = 1$ percentage point; relative to the original, $\frac{1}{4} \times 100\% = 25\%$ increase.

d. $18\% - 15\% = 3$ percentage points; relative to the original, $\frac{3}{15} \times 100\% = 20\%$ increase.

16. Multiply the growth factors—successive percentage changes compound:

a. $\$200 \times 1.15 = \$230$ after the rise; $\$230 \times 0.85 = \$195.50$ after the fall. Overall: $\frac{195.50-200}{200} \times 100\% = -2.25\%$. A 15% rise and a 15% fall do *not* cancel, because the fall applies to the larger amount.

b. $\$50{,}000 \times 1.04^2 = \$50{,}000 \times 1.0816 = \$54{,}080$. The total increase is $1.04^2 - 1 = 8.16\%$, a touch more than $4\% + 4\%$ thanks to compounding.

17.

a. $7{,}250{,}000 = 7.25 \times 10^6$ (move the decimal point 6 places left).

b. $4.8 \times 10^{-5} = 0.000048$ (move the decimal point 5 places left).

c. $(3 \times 10^4)(2 \times 10^3) = (3 \times 2) \times 10^{4+3} = 6 \times 10^7$.

d. $\frac{2.5\times10^{12}}{5\times10^7} = \frac{2.5}{5} \times 10^{12-7} = 0.5 \times 10^5 = 5 \times 10^4$, i.e. \$50,000 per person.

Chapter 2: Powers, Roots, and Logarithms

1. Combine the exponents first, then calculate:

a. $5^2 \cdot 5^4 = 5^{2+4} = 5^6 = 15{,}625$

b. $\dfrac{7^5}{7^2} = 7^{5-2} = 7^3 = 343$

c. $(3^2)^4 = 3^{2 \cdot 4} = 3^8 = 6{,}561$

d. $\dfrac{(4^3)^2}{4^4} = \dfrac{4^6}{4^4} = 4^{6-4} = 4^2 = 16$

2. When the bases match, add or subtract the exponents:

a. $2^3 \times 2^4 = 2^{3+4} = 2^7 = 128$

b. $\dfrac{5^6}{5^2} = 5^{6-2} = 5^4 = 625$

c. $(3^2)^3 \times 3^{-1} = 3^6 \times 3^{-1} = 3^{6-1} = 3^5 = 243$

d. $\dfrac{(2^3 \times 2^{-1})^2}{2^4} = \dfrac{(2^{3-1})^2}{2^4} = \dfrac{(2^2)^2}{2^4} = \dfrac{2^4}{2^4} = 1$

e. Write everything as powers of 2:

$$\frac{4^3 \times 8^{-2}}{2^{-1}} = \frac{(2^2)^3 \times (2^3)^{-2}}{2^{-1}} = \frac{2^6 \times 2^{-6}}{2^{-1}} = \frac{2^0}{2^{-1}} = 2^{0-(-1)} = 2$$

3. For a root, find the number that, raised to the index, gives the radicand:

a. $\sqrt{196} = 14$, since $14^2 = 196$.

b. $\sqrt[3]{512} = 8$, since $8^3 = 512$.

c. $\sqrt[4]{625} = 5$, since $5^4 = 625$.

d. $\sqrt{225x^2} = \sqrt{225}\,\sqrt{x^2} = 15|x|$—the absolute value keeps the result nonnegative when $x < 0$.

e. $32^{\frac{3}{5}} = (32^{\frac{1}{5}})^3 = 2^3 = 8$, since $2^5 = 32$.

f. $(125^2)^{\frac{1}{3}} = (125^{\frac{1}{3}})^2 = 5^2 = 25$, since $5^3 = 125$.

4. Take the appropriate root of both sides; even powers give two solutions:

a. $x^2 = 169 \;\Rightarrow\; x = \pm\sqrt{169} = \pm 13$.

b. $x^3 = 343 \;\Rightarrow\; x = \sqrt[3]{343} = 7$.

c. $x^4 = 81 \;\Rightarrow\; x = \pm\sqrt[4]{81} = \pm 3$.

5. Take the root first, then apply the remaining power:

a. $16^{\frac{1}{2}} = \sqrt{16} = 4$

b. $27^{\frac{1}{3}} = \sqrt[3]{27} = 3$

c. $81^{\frac{3}{4}} = (81^{\frac{1}{4}})^3 = 3^3 = 27$

d. $32^{\frac{2}{5}} = (32^{\frac{1}{5}})^2 = 2^2 = 4$

e. $64^{-\frac{1}{2}} = \dfrac{1}{64^{\frac{1}{2}}} = \dfrac{1}{8}$

6. Ask: to what power must the base be raised to get the argument?

a. $\log_2(128) = 7$, since $2^7 = 128$.

b. $\log_3(243) = 5$, since $3^5 = 243$.

c. $\log_{10}(0.001) = -3$, since $10^{-3} = 0.001$.

d. $\log_5(1) = 0$, since $5^0 = 1$—the log of 1 is always 0.

7. Combine the logs first, using $\log(ab) = \log a + \log b$ and $\log(a/b) = \log a - \log b$:

a. $\log_2(8) + \log_2(4) = \log_2(8 \cdot 4) = \log_2(32) = 5$

b. $\log_3(81) - \log_3(3) = \log_3\left(\dfrac{81}{3}\right) = \log_3(27) = 3$

c. $2\log_5(25) = \log_5(25^2) = \log_5(625) = 4$—or simply $2 \cdot 2 = 4$, since $\log_5(25) = 2$.

d. $\log_{10}(1000) + \log_{10}(0.1) = \log_{10}(1000 \cdot 0.1) = \log_{10}(100) = 2$

8.

a. $\log_2(8) = 3$, since $2^3 = 8$.

b. $\log_3(81) = 4$, since $3^4 = 81$.

c. $\log_5\left(\dfrac{1}{25}\right) = -2$, since $5^{-2} = \dfrac{1}{25}$.

d. $\log_2(32) + \log_2(4) = \log_2(128) = 7$

e. $\log_3(81) - \log_3(9) = \log_3\left(\dfrac{81}{9}\right) = \log_3(9) = 2$

9. Use $\log_b(x) = \dfrac{\ln x}{\ln b}$; in practice, look for how the base and argument are related as powers:

a. $\log_2(8) - \log_4(16) = 3 - 2 = 1$, since $4^2 = 16$.

b. $\log_9(27) = \frac{3}{2}$, since $9^{3/2} = (9^{1/2})^3 = 3^3 = 27$; adding $\log_3(9) = 2$ gives $\frac{3}{2} + 2 = \frac{7}{2}$.

c. $\log_8(4) = \frac{2}{3}$, since $8^{2/3} = (8^{1/3})^2 = 2^2 = 4$; and $\log_4(2) = \frac{1}{2}$, since $4^{1/2} = 2$. So $\frac{2}{3} + \frac{1}{2} = \frac{7}{6}$.

d. $\log_{25}(125) = \frac{3}{2}$, since $25^{3/2} = (25^{1/2})^3 = 5^3 = 125$; and $\log_5(25) = 2$. So $\frac{3}{2} - 2 = -\frac{1}{2}$.

e. $\log_4(64) = 3$, since $4^3 = 64$; and $\log_8(2) = \frac{1}{3}$, since $8^{1/3} = 2$. So $3 + \frac{1}{3} = \frac{10}{3}$.

10.

a. $\dfrac{(2^5 \cdot 2^3)^2}{2^{10}} = \dfrac{(2^8)^2}{2^{10}} = \dfrac{2^{16}}{2^{10}} = 2^6 = 64$

b. $\dfrac{16^3}{4^2} = \dfrac{(2^4)^3}{(2^2)^2} = \dfrac{2^{12}}{2^4} = 2^8 = 256$

c. $\log_2\left(\dfrac{64 \cdot 8}{4}\right) = \log_2(128) = 7$

d. $\sqrt[3]{125}\,\log_5(625) = 5 \cdot 4 = 20$, since $\sqrt[3]{125} = 5$ and $5^4 = 625$.

11.

a. $2^3 \cdot 4^{\frac{1}{2}} = 8 \cdot 2 = 16$

b. $81^{\frac{1}{4}} \cdot 3^2 = 3 \cdot 9 = 27$

c. $\log_2(32) + \log_2(4) - \log_2(8) = \log_2\left(\dfrac{32 \cdot 4}{8}\right) = \log_2(16) = 4$

d. $\log_3(81^{\frac{1}{2}}) = \log_3(9) = 2$

e. $\dfrac{16^{\frac{3}{4}}}{2^2} + \log_4(64) = \dfrac{8}{4} + 3 = 2 + 3 = 5$

12. The approximation $\ln(1 + g) \approx g$ works well only when g is small:

a. $g = 0.50$: $\ln(1.5) \approx 0.4055$ versus 0.50—not accurate (the gap is about 0.095).

b. $g = 0.02$: $\ln(1.02) \approx 0.0198$ versus 0.02—accurate (they agree to three decimals).

c. $g = -0.50$: $\ln(0.5) \approx -0.6931$ versus -0.50—not accurate (the gap is about 0.19).

d. $g = 0.10$: $\ln(1.1) \approx 0.0953$ versus 0.10—reasonably accurate (the gap is under 0.005).

e. $g = -0.01$: $\ln(0.99) \approx -0.0101$ versus -0.01—accurate (they agree to three decimals).

Chapter 3: Algebraic Foundations for Economic Analysis

1. Combine like terms (terms with the same variable part):

a. $3x + 5x - 2x = (3 + 5 - 2)x = 6x$

b. $4y - 3 + 2y + 7 = (4y + 2y) + (-3 + 7) = 6y + 4$

c. $2a + 3b - a + 5b = (2a - a) + (3b + 5b) = a + 8b$

d. Distribute first, then combine: $5(2x - 3) - 2(x + 4) = 10x - 15 - 2x - 8 = 8x - 23$

e. $3(2x + y) - 2(x - 4y) = 6x + 3y - 2x + 8y = 4x + 11y$

f. $6m^2 - 2m + 3m^2 + 7m = (6m^2 + 3m^2) + (-2m + 7m) = 9m^2 + 5m$

g. $12q + 7q - 4(3q + 2) = 19q - 12q - 8 = 7q - 8$

2. Multiply out, then collect like terms:

a. $(3 - x)^2 = 9 - 6x + x^2 = x^2 - 6x + 9$

b. $(y + 5)^2 = y^2 + 10y + 25$

c. $(x - 4)(x + 4) = x^2 - 16$ (difference of squares)

d. Expand each piece:

$$\begin{aligned}(1 - 2x)^2 + (y - 3)^2 + x(y + 2) - 2xy &= (1 - 4x + 4x^2) + (y^2 - 6y + 9) + (xy + 2x) - 2xy \\ &= 4x^2 + y^2 - xy - 2x - 6y + 10.\end{aligned}$$

3. Expand and cancel:

a. $(x + 2)^2 - (x - 2)^2 = (x^2 + 4x + 4) - (x^2 - 4x + 4) = 8x$

b. $(x+y)^2 - (x-y)^2 = (x^2+2xy+y^2) - (x^2-2xy+y^2) = 4xy$

4. Pull out the common factor first, then factor further if you can:

a. $8x^2 + 12x = 4x(2x+3)$

b. $Q^2 - 49 = (Q-7)(Q+7)$ (difference of squares)

c. $x^2 + 7x + 12 = (x+3)(x+4)$, since $3+4=7$ and $3 \cdot 4 = 12$

d. $2y^2 - 18 = 2(y^2-9) = 2(y-3)(y+3)$

5. Isolate the variable:

a. $3x + 5 = 20 \;\Rightarrow\; 3x = 15 \;\Rightarrow\; x = 5$

b. $4p - 7 = 21 \;\Rightarrow\; 4p = 28 \;\Rightarrow\; p = 7$

c. $\frac{y}{5} + 2 = 8 \;\Rightarrow\; \frac{y}{5} = 6 \;\Rightarrow\; y = 30$

d. $2(a-3) = 10 \;\Rightarrow\; a - 3 = 5 \;\Rightarrow\; a = 8$

e. $P = 100 - 4q \;\Rightarrow\; 4q = 100 - P \;\Rightarrow\; q = \dfrac{100-P}{4}$

6. Plug $q = 400$ and $I = 60$ into the demand equation:

$$400 = 200 - 4p + 5(60) = 200 - 4p + 300 = 500 - 4p.$$

So $4p = 500 - 400 = 100$, giving $p = 25$. The shop charges \$25 per iced coffee.

7. Solve each inequality, remembering to flip the sign when you multiply or divide by a negative:

a. $3x + 4 > 10 \;\Rightarrow\; 3x > 6 \;\Rightarrow\; x > 2$

b. $5p - 7 \leq 18 \;\Rightarrow\; 5p \leq 25 \;\Rightarrow\; p \leq 5$

c. $\frac{y}{4} + 3 < 8 \;\Rightarrow\; \frac{y}{4} < 5 \;\Rightarrow\; y < 20$

d. $2(a-1) \geq 12 \;\Rightarrow\; a - 1 \geq 6 \;\Rightarrow\; a \geq 7$

e. $-3q + 6 > 15 \;\Rightarrow\; -3q > 9 \;\Rightarrow\; q < -3$ (sign flips)

8. Track the numerator and denominator as the variable grows:

a. $\dfrac{p^2+1}{7}$: the numerator p^2+1 grows for $p \geq 0$ while the denominator is constant, so the fraction **increases**.

b. $\dfrac{5}{\sqrt{q}+2}$: the denominator $\sqrt{q}+2$ grows while the numerator is fixed, so the fraction **decreases**.

c. $\dfrac{\ln(r)+4}{6}$: $\ln r$ grows with r, so the fraction **increases**.

d. $\dfrac{2s^2+3}{8}$: the numerator grows for $s \geq 0$, so the fraction **increases**.

e. $\dfrac{5}{t^2+4}$: the denominator grows while the numerator is fixed, so the fraction **decreases**.

9.

a. $\dfrac{5}{x}$: the denominator grows, so the fraction **decreases**.

b. $2x+7$: each extra unit of x adds 2, so it **increases**.

c. $-x+\dfrac{3}{x}$: both $-x$ and $3/x$ fall as x grows, so the sum **decreases**.

d. $\dfrac{50-x}{x+10}$: rewrite it as $\dfrac{60-(x+10)}{x+10} = \dfrac{60}{x+10} - 1$. As x grows, $\dfrac{60}{x+10}$ falls, so the expression **decreases**.

10. Use $|u| = u$ when $u \geq 0$ and $|u| = -u$ when $u < 0$:

a. $|2x-6| = \begin{cases} 2x-6, & x \geq 3 \\ 6-2x, & x < 3 \end{cases}$

b. $|p+4| = \begin{cases} p+4, & p \geq -4 \\ -p-4, & p < -4 \end{cases}$

c. $|5-q| = \begin{cases} 5-q, & q \leq 5 \\ q-5, & q > 5 \end{cases}$

d. $|r^2-9| = \begin{cases} r^2-9, & |r| \geq 3 \\ 9-r^2, & |r| < 3 \end{cases}$

e. $|3s+12| = \begin{cases} 3s+12, & s \geq -4 \\ -3s-12, & s < -4 \end{cases}$

11. Split each into two cases:

a. $|x| = 12 \;\Rightarrow\; x = 12$ or $x = -12$, i.e. $x = \pm 12$.

b. $|3x-1| = 8 \;\Rightarrow\; 3x-1 = 8$ or $3x-1 = -8$, so $x = 3$ or $x = -\frac{7}{3}$.

12. Unpack each absolute value into an interval (or two):

a. $|x-3| < 5 \;\Rightarrow\; -5 < x-3 < 5 \;\Rightarrow\; -2 < x < 8$

b. $|2p+1| \leq 7 \;\Rightarrow\; -7 \leq 2p+1 \leq 7 \;\Rightarrow\; -8 \leq 2p \leq 6 \;\Rightarrow\; -4 \leq p \leq 3$

c. $|y-4| > 3 \;\Rightarrow\; y-4 > 3$ or $y-4 < -3 \;\Rightarrow\; y > 7$ or $y < 1$

d. $|3a-6| \geq 12 \;\Rightarrow\; 3a-6 \geq 12$ or $3a-6 \leq -12 \;\Rightarrow\; a \geq 6$ or $a \leq -2$

e. $|q+2| < 4 \;\Rightarrow\; -4 < q+2 < 4 \;\Rightarrow\; -6 < q < 2$

Chapter 4: Algebra and Economic Modeling

1.

a. Each pound needs 0.2 square feet, and land rents for \$2 per square foot, so the land rental cost is
$$0.2q \times 2 = 0.4q \quad \text{dollars.}$$

b. The capacity-pressure cost is given directly in the problem: $\frac{q^2}{500}$ dollars.

c. Total variable cost adds the two variable pieces:
$$TVC(q) = 0.4q + \frac{q^2}{500}.$$

d. Total cost adds the fixed \$120 market fee:
$$TC(q) = 120 + 0.4q + \frac{q^2}{500}.$$

2. Plug into $TC(q) = 120 + 0.4q + q^2/500$:

a. $TC(400) = 120 + 0.4(400) + \dfrac{400^2}{500} = 120 + 160 + 320 = 600.$

b. $TC(900) = 120 + 0.4(900) + \dfrac{900^2}{500} = 120 + 360 + 1620 = 2100.$

3.

a. Revenue is price times quantity: $R(q) = 3q$.

b. Profit is revenue minus cost:

$$\pi(q) = 3q - \left(120 + 0.4q + \frac{q^2}{500}\right) = -\frac{q^2}{500} + 2.6q - 120.$$

c. Complete the square. Factor $-\frac{1}{500}$ out of the q-terms, then add and subtract $(1300/2)^2 = 650^2$ inside:

$$\begin{aligned}\pi(q) &= -\frac{1}{500}\left(q^2 - 1300q\right) - 120\\ &= -\frac{(q-650)^2}{500} + \frac{650^2}{500} - 120\\ &= 725 - \frac{(q-650)^2}{500}.\end{aligned}$$

The squared term is always nonnegative and is subtracted, so profit is largest when it equals zero: $q^* = 650$, with maximum profit \$725.

4.

a. Apples contribute \$2.50 each and oranges \$1.80 each:

$$C(x, y) = 2.5x + 1.8y.$$

b. $C(12, 8) = 2.5(12) + 1.8(8) = 30 + 14.4 = 44.4$, i.e. \$44.40.

c. $C(20, 15) = 2.5(20) + 1.8(15) = 50 + 27 = 77$, i.e. \$77.

5. Substitute into $C(q) = 500 + 2.5q + q^2/200$:

a. $C(100) = 500 + 2.5(100) + \dfrac{100^2}{200} = 500 + 250 + 50 = 800.$

b. $C(400) = 500 + 2.5(400) + \dfrac{400^2}{200} = 500 + 1000 + 800 = 2300.$

6.

a. $\pi(q) = R(q) - C(q) = 6q - \left(300 + 3q + \dfrac{q^2}{200}\right) = -\dfrac{q^2}{200} + 3q - 300.$

b. $\pi(100) = -\dfrac{100^2}{200} + 3(100) - 300 = -50 + 300 - 300 = -50$, a loss of \$50.

c. Completing the square as in Question 3:

$$\pi(q) = -\frac{1}{200}\left(q^2 - 600q\right) - 300 = -\frac{(q-300)^2}{200} + \frac{300^2}{200} - 300 = 150 - \frac{(q-300)^2}{200}.$$

Profit is maximized at $q^* = 300$, with maximum profit \$150.

7.

a. Potatoes need 0.1 sq ft per pound and tomatoes 0.15 sq ft per pound, so the total land requirement is $0.1p + 0.15t$ square feet.

b. At \$3 per square foot, the rental cost is $3(0.1p + 0.15t) = 0.3p + 0.45t$ dollars.

c. $0.3(800) + 0.45(600) = 240 + 270 = 510$, i.e. \$510.

8.

a. Each candle uses 0.5 pounds of wax, so q candles need $0.5q$ pounds.

b. At \$4 per pound, the wax cost is $4(0.5q) = 2q$ dollars.

c. Each box holds 10 candles, so q candles need $q/10$ boxes at \$3 each: $3(q/10) = 0.3q$ dollars.

d. Add wax and packaging: $TVC(q) = 2q + 0.3q = 2.3q$.

e. Add the fixed booth fee: $TC(q) = 150 + 2.3q$.

f. At \$8 per candle: $R(q) = 8q$.

g. $\Pi(q) = R(q) - TC(q) = 8q - (150 + 2.3q) = 5.7q - 150.$

9.

a. Flour: $0.2m + 0.1c$ pounds.

b. Sugar: $0.1m + 0.05c$ pounds.

c. Flour costs \$2 per pound: $2(0.2m + 0.1c) = 0.4m + 0.2c$ dollars.

d. Sugar costs \$1.50 per pound: $1.5(0.1m + 0.05c) = 0.15m + 0.075c$ dollars.

e. Add the two: $TVC(m, c) = (0.4m + 0.2c) + (0.15m + 0.075c) = 0.55m + 0.275c$.

f. Add the \$300 rent: $TC(m, c) = 300 + 0.55m + 0.275c$.

g. Muffins sell for \$4 and cookies for \$2: $R(m, c) = 4m + 2c$.

h. $\Pi(m, c) = (4m + 2c) - (300 + 0.55m + 0.275c) = 3.45m + 1.725c - 300$.

10.

a. Demand falls by 40 tickets: the effect is -40.

b. Demand falls by 20 tickets: the effect is -20.

c. Start from 1200 tickets at zero prices and subtract each price effect:

$$D(p_A, p_C) = 1200 - 40p_A - 20p_C.$$

11.

a. Demand drops 150 per \$1,000 of price, starting from 15,000 at $p = 0$:

$$D(p) = 15000 - 150p.$$

b. Consumers pay the sticker price plus the tariff: $p + t$ thousand dollars.

c. Replace p by $p + t$ in the demand function:

$$D(p, t) = 15000 - 150(p + t).$$

d. Here $p = 30$ and $t = 4$, so consumers face 34:

$$D = 15000 - 150(34) = 15000 - 5100 = 9900 \text{ vehicles per month.}$$

12. Substitute into $P = 120 - 2q_P - 3q_C - q_B$:

a. $P = 120 - 2(10) - 3(8) - 12 = 120 - 20 - 24 - 12 = 64$ dollars per barrel.

b. With $q_B = 0$: $P = 120 - 2(10) - 3(8) - 0 = 120 - 20 - 24 = 76$ dollars per barrel. Less supply means a higher price.

13.

a. Value starts at \$24,000 and falls \$2,500 per year:

$$V(t) = 24000 - 2500t.$$

b. $V(2) = 24000 - 2500(2) = 24000 - 5000 = 19000$.

c. $V(4) = 24000 - 2500(4) = 24000 - 10000 = 14000$.

d. $V(6) = 24000 - 2500(6) = 24000 - 15000 = 9000$.

e. Set $V(t) = 11500$ and solve:

$$24000 - 2500t = 11500 \Rightarrow 2500t = 12500 \Rightarrow t = 5 \text{ years}.$$

14. Substitute into $I(a, n, r) = 0.5a + 0.3n + 0.2r$:

a. $0.5(40) + 0.3(30) + 0.2(20) = 20 + 9 + 4 = 33$.

b. $0.5(50) + 0.3(28) + 0.2(18) = 25 + 8.4 + 3.6 = 37$.

c. $0.5(36) + 0.3(48) + 0.2(20) = 18 + 14.4 + 4 = 36.4$.

d. $0.5(42) + 0.3(31) + 0.2(10) = 21 + 9.3 + 2 = 32.3$.

e. $0.5(45) + 0.3(55) + 0.2(25) = 22.5 + 16.5 + 5 = 44$.

Chapter 5: Functions and Linear Relationships in Economics

1. The domain is where the formula makes sense; the range is what comes out.

a. $f(x) = \sqrt{x+4}$ needs $x + 4 \geq 0$, so the domain is $x \geq -4$. A square root is never negative, so the range is $f(x) \geq 0$.

b. $g(x) = \dfrac{1}{x+2}$ is undefined when the denominator is zero, so the domain is $x \neq -2$. The fraction can equal any nonzero number, so the range is $y \neq 0$.

c. $Q = 200 - 5P$ with $P \geq 0$ and $Q \geq 0$. From $Q \geq 0$: $200 - 5P \geq 0$, so $P \leq 40$; hence the domain is $0 \leq P \leq 40$. When $P = 0$, $Q = 200$, and when $P = 40$, $Q = 0$, so the range is $0 \leq Q \leq 200$.

2. Just substitute the given value for x:

a. $f(3) = 2(3) + 1 = 7$

b. $f(-2) = -(-2) + 4 = 6$

c. $f(4) = 3(4) - 5 = 7$

d. $f(8) = \frac{1}{2}(8) + 2 = 6$

e. $f(5) = -2(5) + 7 = -3$

3. Plug the point's x-value into the function and see whether you get its y-value:

a. $2(3) + 1 = 7$: yes, $(3, 7)$ lies on the graph.

b. $-(2) + 4 = 2 \neq 1$: no.

c. $3(4) - 5 = 7 \neq 8$: no.

d. $\frac{1}{2}(6) + 2 = 5$: yes, $(6, 5)$ lies on the graph.

e. $-2(3) + 7 = 1 \neq 2$: no.

4. For each line, plot the y-intercept (set $x = 0$) and the x-intercept (set $y = 0$), then draw the straight line through them:

a. $y = 2x + 1$: y-intercept $(0, 1)$; x-intercept $(-\frac{1}{2}, 0)$.

b. $y = -x + 3$: y-intercept $(0, 3)$; x-intercept $(3, 0)$.

c. $y = \frac{1}{2}x - 2$: y-intercept $(0, -2)$; x-intercept $(4, 0)$.

d. $y = -2x + 4$: y-intercept $(0, 4)$; x-intercept $(2, 0)$.

e. $y = 3x - 3$: y-intercept $(0, -3)$; x-intercept $(1, 0)$.

5. Read the y-intercept off the graph (where the line crosses the vertical axis), then count rise over run for the slope:

a. Crosses at 1 and rises 2 for each unit right: $y = 2x + 1$.

b. Crosses at 3 and falls 1 for each unit right: $y = -x + 3$.

c. Crosses at -2 and rises 1 for every 2 units right: $y = \frac{1}{2}x - 2$.

d. Crosses at 4 and falls 2 for each unit right: $y = -2x + 4$.

e. Crosses at -3 and rises 3 for each unit right: $y = 3x - 3$.

6. Slope $m = \dfrac{y_2 - y_1}{x_2 - x_1}$:

a. $m = \dfrac{9-3}{4-1} = \dfrac{6}{3} = 2$

b. $m = \dfrac{0-5}{3-(-2)} = \dfrac{-5}{5} = -1$

c. $m = \dfrac{7-(-1)}{6-2} = \dfrac{8}{4} = 2$

d. $m = \dfrac{6-(-3)}{2-(-4)} = \dfrac{9}{6} = \dfrac{3}{2}$

e. $m = \dfrac{2-4}{5-0} = \dfrac{-2}{5} = -\dfrac{2}{5}$

7. Compute $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, then use $y - y_1 = m(x - x_1)$:

a. $m = \dfrac{11-5}{5-2} = 2$. Then $y - 5 = 2(x-2)$, so $y = 2x + 1$.

b. $m = \dfrac{-4-4}{3-(-1)} = -2$. Then $y - 4 = -2(x+1)$, so $y = -2x + 2$.

c. $m = \dfrac{6-(-2)}{4-0} = 2$. Since the line passes through $(0, -2)$, $y = 2x - 2$.

d. $m = \dfrac{1-7}{1-(-3)} = -\dfrac{3}{2}$. Then $y - 1 = -\dfrac{3}{2}(x-1)$, so $y = -\dfrac{3}{2}x + \dfrac{5}{2}$. (Check: at $x = -3$, $y = \frac{9}{2} + \frac{5}{2} = 7$; at $x = 1$, $y = -\frac{3}{2} + \frac{5}{2} = 1$.)

e. $m = \dfrac{2-(-3)}{7-2} = 1$. Then $y + 3 = 1(x-2)$, so $y = x - 5$.

8. The x-intercept is where the line crosses the horizontal axis ($y = 0$); the y-intercept is where it crosses the vertical axis ($x = 0$):

a. $y = 2x + 2$: x-intercept $(-1, 0)$, y-intercept $(0, 2)$

b. $y = -x + 3$: x-intercept $(3, 0)$, y-intercept $(0, 3)$

c. $y = \frac{1}{2}x - 2$: x-intercept $(4, 0)$, y-intercept $(0, -2)$

d. $y = -2x - 4$: x-intercept $(-2, 0)$, y-intercept $(0, -4)$

e. $y = 3x - 3$: x-intercept $(1, 0)$, y-intercept $(0, -3)$

9.

a. The coefficient of x is the slope: $-\dfrac{3}{4}$.

b. The constant term is the y-intercept: 12.

c. $y = -\dfrac{3}{4}(8) + 12 = -6 + 12 = 6$.

d. $(0, 12)$ (the intercept) and $(8, 6)$ from part (c) both lie on the line.

e. Multiply through to clear the fraction: $\dfrac{3}{4}x + y - 12 = 0$, i.e. $3x + 4y - 48 = 0$.

10.

a. The intercepts give $\dfrac{x}{8} + \dfrac{y}{12} = 1$.

b. Solve for y: $\dfrac{y}{12} = 1 - \dfrac{x}{8}$, so $y = -\dfrac{3}{2}x + 12$.

c. The slope is $-\dfrac{3}{2}$.

d. The slope is negative, so the line is decreasing.

11. Start from $\dfrac{x}{5} + \dfrac{y}{20} = 1$ and multiply by 20 to clear denominators: $4x + y = 20$.

a. Set $y = 0$: $4x = 20$, so the x-intercept is 5.

b. Set $x = 0$: $y = 20$, so the y-intercept is 20.

c. Solve for y: $y = -4x + 20$.

d. The slope is -4.

12.

a. The year is what you choose to look at: it is the independent variable.

b. Profit depends on the year: it is the dependent variable.

c. $P(2023) = 25$ (million dollars).

d. $P(2025) = 30$ (million dollars).

13.

a. Yes: it has the form $C = aY + b$ with $a = 0.75$ and $b = 6000$.

b. $C(20{,}000) = 0.75(20{,}000) + 6000 = 15{,}000 + 6000 = 21{,}000$.

c. $C(40{,}000) = 0.75(40{,}000) + 6000 = 30{,}000 + 6000 = 36{,}000$.

d. $36{,}000 - 21{,}000 = 15{,}000$ dollars.

e. The marginal propensity to consume: each extra dollar of income raises consumption by 75 cents.

14.

a. $Q = 500 - 8(20) = 500 - 160 = 340$ tickets.

b. $Q = 500 - 8(35) = 500 - 280 = 220$ tickets.

c. $8p = 500 - Q$, so $p = 62.5 - \dfrac{Q}{8}$.

d. Demand falls: a higher price means fewer tickets sold.

15.

a. $m = \dfrac{95 - 50}{25 - 10} = \dfrac{45}{15} = 3$.

b. Each additional unit sold raises revenue by \$3,000 (revenue is measured in thousands of dollars).

16.

a. The fixed fee plus the per-mile charge: $F(m) = 2.5m + 4$.

b. Yes; with $F(m) = am + b$, you read off $a = 2.5$ and $b = 4$.

c. $F(12) = 2.5(12) + 4 = 34$ dollars; $F(20) = 2.5(20) + 4 = 54$ dollars.

d. $54 - 34 = 20$ dollars.

17.

a. In 2024, $x = 4$: $A(4) = 5(4) + 10 = 30$ and $B(4) = 3(4) + 20 = 32$ (million dollars).

b. Company A: its slope 5 exceeds company B's slope 3.

c. Company A's profit grows by \$5 million per year; company B's by \$3 million per year.

d. Set them equal: $5x + 10 = 3x + 20$, so $2x = 10$ and $x = 5$, which is the year 2025.

e. In 2030, $x = 10$: $A(10) = 60$ and $B(10) = 50$, so company A has the larger profit.

18.

a. The slope of Q_A is -5; the slope of Q_B is -2.

b. Product A: its demand curve is steeper, so each dollar of price increase cuts demand by more.

c. At $p = 20$: $Q_A = 400 - 5(20) = 300$ and $Q_B = 400 - 2(20) = 360$.

d. Product B has the greater demand at $p = 20$.

19.

a. $m = \dfrac{7000 - 3000}{300 - 100} = \dfrac{4000}{200} = 20$.

b. Using $(100, 3000)$: $R - 3000 = 20(Q - 100)$, so $R(Q) = 20Q + 1000$.

c. $R(500) = 20(500) + 1000 = 11{,}000$ (dollars).

20.

a. Spending on books plus spending on notebooks equals \$120: $10x + 6y = 120$.

b. Solve for y: $6y = 120 - 10x$, so $y = -\frac{5}{3}x + 20$.

c. The slope is $-\frac{5}{3}$.

d. The relative price of books in terms of notebooks: one more book costs $\frac{5}{3}$ notebooks.

e. x-intercept: set $y = 0$, giving $10x = 120$, so $x = 12$ books. y-intercept: set $x = 0$, giving $6y = 120$, so $y = 20$ notebooks.

Chapter 6: Systems of Linear Equations

1. Both equations are already solved for y, so set the right-hand sides equal:

a. $4x - 7 = -2x + 17 \;\Rightarrow\; 6x = 24 \;\Rightarrow\; x = 4$. Then $y = 4(4) - 7 = 9$. Check: $-2(4) + 17 = 9$. ✓

b. $5x + 1 = -3x + 25 \;\Rightarrow\; 8x = 24 \;\Rightarrow\; x = 3$. Then $y = 5(3) + 1 = 16$. Check: $-3(3) + 25 = 16$. ✓

2.

a. Substitute $x = 2y - 1$ into the first equation:

$$\begin{aligned}(2y - 1) + 2y &= 19\\ 4y &= 20 \;\Rightarrow\; y = 5,\end{aligned}$$

so $x = 2(5) - 1 = 9$. Check: $9 + 2(5) = 19$. ✓

b. Substitute $y = \frac{x}{2} + 4$ into $3x - 2y = 7$:

$$\begin{aligned}3x - 2\left(\frac{x}{2} + 4\right) &= 7\\ 3x - x - 8 &= 7 \;\Rightarrow\; 2x = 15 \;\Rightarrow\; x = \frac{15}{2},\end{aligned}$$

so $y = \frac{15}{4} + 4 = \frac{31}{4}$. Check: $3\left(\frac{15}{2}\right) - 2\left(\frac{31}{4}\right) = \frac{45-31}{2} = 7$. ✓

c. Eliminate y: multiply the first equation by 3 and the second by 2,

$$\begin{aligned} 21x + 12y &= 138 \\ 10x - 12y &= -4, \end{aligned}$$

and add: $31x = 134$, so $x = \frac{134}{31}$. Then $4y = 46 - 7\left(\frac{134}{31}\right) = \frac{488}{31}$, giving $y = \frac{122}{31}$. Check: $5\left(\frac{134}{31}\right) - 6\left(\frac{122}{31}\right) = \frac{670-732}{31} = -2$. ✓

3.

a. Add the equations to eliminate y: $3x = 6$, so $x = 2$ and $y = 5 - 2 = 3$. Check: $2(2) - 3 = 1$. ✓

b. Multiply the second equation by 2 to get $6x - 2y = 8$, then add to the first:

$$\begin{aligned} x + 2y &= 7 \\ 6x - 2y &= 8, \end{aligned}$$

so $7x = 15$ and $x = \frac{15}{7}$. Then $y = 3\left(\frac{15}{7}\right) - 4 = \frac{17}{7}$. Check: $\frac{15}{7} + 2\left(\frac{17}{7}\right) = \frac{49}{7} = 7$. ✓

c. Add the equations to eliminate y: $3x = 9$, so $x = 3$ and $y = 3 - 1 = 2$. Check: $2(3) + 2 = 8$. ✓

d. From the second equation, $y = 8 - 2x$; substitute into the first:

$$\begin{aligned} x + 3(8 - 2x) &= 10 \\ x + 24 - 6x &= 10 \;\Rightarrow\; -5x = -14 \;\Rightarrow\; x = \frac{14}{5}, \end{aligned}$$

so $y = 8 - 2\left(\frac{14}{5}\right) = \frac{12}{5}$. Check: $\frac{14}{5} + 3\left(\frac{12}{5}\right) = \frac{50}{5} = 10$. ✓

e. From the second equation, $x = 7 - 2y$; substitute into the first:

$$\begin{aligned} 3(7 - 2y) + y &= 11 \\ 21 - 5y &= 11 \;\Rightarrow\; y = 2, \end{aligned}$$

so $x = 7 - 4 = 3$. Check: $3 + 2(2) = 7$. ✓

4.

a. Add the first and third equations to eliminate z: $2x + 3y = 8$. Add the second and third: $3x + y = 5$, so $y = 5 - 3x$. Substitute:

$$2x + 3(5 - 3x) = 8 \Rightarrow -7x = -7 \Rightarrow x = 1,$$

then $y = 5-3 = 2$ and $z = 6-1-2 = 3$. Check in the second equation: $2(1)-2+3 = 3$. ✓

b. Add the first and third equations: $4x+3y = 19$. Add the second and third: $5x+y = 21$, so $y = 21 - 5x$. Substitute:

$$4x + 3(21 - 5x) = 19 \Rightarrow -11x = -44 \Rightarrow x = 4,$$

then $y = 21 - 20 = 1$ and $z = 7 - 4 - 1 = 2$. Check in the second equation: $2(4) - 1 + 2 = 9$. ✓

5.

a. Add the equations: $3x = 6$, so $x = 2$, $y = 3$. Unique solution $(2, 3)$.

b. The second equation is twice the first, so both say the same thing: infinitely many solutions (every point on the line $x + 2y = 6$).

c. Twice the first equation gives $2x-2y = 6$, which contradicts $2x-2y = 10$: no solution.

d. The second equation is twice the first: infinitely many solutions.

e. Twice the first equation gives $4x+6y = 16$, which contradicts $4x+6y = 15$: no solution.

6.

a. The second equation is twice the first: infinitely many solutions.

b. Twice the first equation gives $6x - 2y = 10$, contradicting $6x - 2y = 8$: no solution.

c. $2x + 1 = -x + 7 \Rightarrow 3x = 6 \Rightarrow x = 2$, and $y = 2(2) + 1 = 5$. Unique solution $(2, 5)$.

7. Set $Q_D = Q_S$ and solve for p, then plug back for Q:

a. $1200 - 40p = 300 + 20p \Rightarrow 900 = 60p \Rightarrow p = 15$; $Q = 1200 - 40(15) = 600$.

b. $1500 - 50p = 200 + 30p \Rightarrow 1300 = 80p \Rightarrow p = 16.25$; $Q = 1500 - 50(16.25) = 687.5$.

c. $900 - 25p = 100 + 15p \;\Rightarrow\; 800 = 40p \;\Rightarrow\; p = 20$; $Q = 900 - 25(20) = 400$.

d. $1800 - 60p = 600 + 20p \;\Rightarrow\; 1200 = 80p \;\Rightarrow\; p = 15$; $Q = 1800 - 60(15) = 900$.

e. $1000 - 20p = 100 + 10p \;\Rightarrow\; 900 = 30p \;\Rightarrow\; p = 30$; $Q = 1000 - 20(30) = 400$.

8. Substitute the demand quantity into the inverse supply, $p = a + bQ_D(p)$, and solve for p:

a. $p = 10 + 0.05(1000 - 20p) = 60 - p \;\Rightarrow\; 2p = 60 \;\Rightarrow\; p = 30$; $Q = 1000 - 20(30) = 400$.

b. $p = 8 + 0.04(1200 - 30p) = 56 - 1.2p \;\Rightarrow\; 2.2p = 56 \;\Rightarrow\; p = \frac{280}{11} \approx 25.45$; $Q = 1200 - 30\left(\frac{280}{11}\right) = \frac{4800}{11} \approx 436.36$.

c. $p = 5 + 0.02(800 - 25p) = 21 - 0.5p \;\Rightarrow\; 1.5p = 21 \;\Rightarrow\; p = 14$; $Q = 800 - 25(14) = 450$.

d. $p = 15 + 0.03(1500 - 50p) = 60 - 1.5p \;\Rightarrow\; 2.5p = 60 \;\Rightarrow\; p = 24$; $Q = 1500 - 50(24) = 300$.

e. $p = 12 + 0.06(900 - 10p) = 66 - 0.6p \;\Rightarrow\; 1.6p = 66 \;\Rightarrow\; p = 41.25$; $Q = 900 - 10(41.25) = 487.5$.

9. Plot each demand curve (downward sloping) and supply curve (upward sloping); they cross at the equilibrium, where $Q_D = Q_S$:

a. $100 - 2p = 20 + p \;\Rightarrow\; 3p = 80 \;\Rightarrow\; p = \frac{80}{3} \approx 26.67$; $Q = 100 - 2\left(\frac{80}{3}\right) = \frac{140}{3} \approx 46.67$.

b. $120 - 3p = 30 + p \;\Rightarrow\; 4p = 90 \;\Rightarrow\; p = 22.5$; $Q = 120 - 3(22.5) = 52.5$.

c. $90 - p = 10 + 3p \;\Rightarrow\; 4p = 80 \;\Rightarrow\; p = 20$; $Q = 90 - 20 = 70$.

d. $80 - 2p = 20 + 2p \;\Rightarrow\; 4p = 60 \;\Rightarrow\; p = 15$; $Q = 80 - 2(15) = 50$.

e. $150 - 5p = 30 + p \;\Rightarrow\; 6p = 120 \;\Rightarrow\; p = 20$; $Q = 150 - 5(20) = 50$.

10. Equate demand and supply:

$$2400 - 6P = 3P + 150 \;\Rightarrow\; 2250 = 9P \;\Rightarrow\; P = 250,$$

and $Q = 2400 - 6(250) = 900$ (check: $3(250) + 150 = 900$). ✓

11. Substitute $p_c = p_p + t$ into $Q_D = Q_S$:

a. $1200 - 30(p_p + 10) = 300 + 20p_p \;\Rightarrow\; 600 = 50p_p \;\Rightarrow\; p_p = 12$, $p_c = 22$, $Q = 300 + 20(12) = 540$.

b. $1500 - 50(p_p + 10) = 300 + 20p_p \Rightarrow 700 = 70p_p \Rightarrow p_p = 10,\ p_c = 20,\ Q = 300 + 20(10) = 500.$

c. $900 - 15(p_p + 6) = 150 + 5p_p \Rightarrow 660 = 20p_p \Rightarrow p_p = 33,\ p_c = 39,\ Q = 150 + 5(33) = 315.$

d. $800 - 20(p_p + 8) = 100 + 10p_p \Rightarrow 540 = 30p_p \Rightarrow p_p = 18,\ p_c = 26,\ Q = 100 + 10(18) = 280.$

e. $1000 - 20(p_p + 10) = 200 + 5p_p \Rightarrow 600 = 25p_p \Rightarrow p_p = 24,\ p_c = 34,\ Q = 200 + 5(24) = 320.$

12.

a. After the tax, equilibrium requires $Q_D = Q_S$ together with the wedge:

$$2500 - 5P_c = 3P_p + 400, \qquad P_c = P_p + 20.$$

b. Substitute the wedge into the market-clearing condition:

$$2500 - 5(P_p + 20) = 3P_p + 400 \Rightarrow 2000 = 8P_p \Rightarrow P_p = 250,$$

so $P_c = 250 + 20 = 270$.

c. $P_p = 250$.

d. $Q = 2500 - 5(270) = 1150$ (check: $3(250) + 400 = 1150$). ✓

13. Set $C_1 = C_2$:

a. $200 + 10Q = 500 + 4Q \Rightarrow 6Q = 300 \Rightarrow Q = 50$; $C = 200 + 10(50) = 700$.

b. $300 + 8Q = 700 + 2Q \Rightarrow 6Q = 400 \Rightarrow Q = \frac{200}{3} \approx 66.67$; $C = 300 + 8\left(\frac{200}{3}\right) = \frac{2500}{3} \approx 833.33$.

c. $150 + 12Q = 450 + 6Q \Rightarrow 6Q = 300 \Rightarrow Q = 50$; $C = 150 + 12(50) = 750$.

d. $100 + 15Q = 400 + 5Q \Rightarrow 10Q = 300 \Rightarrow Q = 30$; $C = 100 + 15(30) = 550$.

e. $250 + 9Q = 700 + 4Q \Rightarrow 5Q = 450 \Rightarrow Q = 90$; $C = 250 + 9(90) = 1060$.

14.

a. Let t be the price of a movie ticket and d the price of a drink. Then $2t + 3d = 33$ and $4t + d = 29$.

b. From the second equation, $d = 29 - 4t$; substitute:

$$2t + 3(29 - 4t) = 33$$
$$2t + 87 - 12t = 33 \;\Rightarrow\; -10t = -54 \;\Rightarrow\; t = 5.4,$$

so $d = 29 - 4(5.4) = 7.4$. A ticket costs \$5.40 and a drink costs \$7.40. Check: $2(5.4) + 3(7.4) = 10.8 + 22.2 = 33$. ✓

15.

a. With $P_A = P_B = P$ and $Q_A = Q_B = Q$, the two demand equations become $Q = 600 - 3P$ and $Q = 500 - P$.

b. Equate them: $600 - 3P = 500 - P \;\Rightarrow\; 100 = 2P \;\Rightarrow\; P = 50$.

c. $Q = 600 - 3(50) = 450$ per firm (check: $500 - 50 = 450$). ✓

d. Total market sales: $450 + 450 = 900$.

Chapter 7: Nonlinear and Multivariable Functions

1. Take the root first, then raise to the remaining power:

a. $8^{2/3} = (8^{1/3})^2 = 2^2 = 4$

b. $27^{1/3} = 3$

c. $4^{-1/2} = \dfrac{1}{4^{1/2}} = \dfrac{1}{2}$

d. $(-8)^{2/3} = \left((-8)^{1/3}\right)^2 = (-2)^2 = 4$

e. $16^{3/4} = (16^{1/4})^3 = 2^3 = 8$

2. Read each exponent off the table in the Power Functions section:

a. $r = 0.5$: since $0 < r < 1$, the function is increasing and concave.

b. $r = 2.5$: since $r > 1$, the function is increasing and convex.

c. $r = -1.5$: since $r < 0$, the function is decreasing and convex.

d. $r = 1.0$: this is linear, so its slope never changes—it is both concave and convex, though neither strictly. It is increasing.

e. $r = 0$: since $x^0 = 1$ for $x > 0$, the function is constant—neither increasing nor decreasing, but both concave and convex (neither strictly).

3.

a. The bases match, so add the exponents:

$$5(100)^{0.4}(100)^{0.6} = 5 \cdot 100^{0.4+0.6} = 5 \cdot 100 = 500.$$

b. The bases differ, so evaluate each factor separately:

$$5(200)^{0.4}(150)^{0.6} \approx 5(8.3255)(20.2141) \approx 841.47.$$

4. Solve for x in terms of y:

a. $y = 3x + 8 \;\Rightarrow\; 3x = y - 8 \;\Rightarrow\; x = \dfrac{y-8}{3}$.

b. $y = x^3 - 2 \;\Rightarrow\; x^3 = y + 2 \;\Rightarrow\; x = \sqrt[3]{y+2}$.

c. $y = \dfrac{x+4}{2} \;\Rightarrow\; 2y = x + 4 \;\Rightarrow\; x = 2y - 4$.

d. $y = e^x \;\Rightarrow\; x = \ln y$, defined for $y > 0$.

5.

a. $y = x^2$ falls for $x < 0$ and rises for $x > 0$, so it is neither increasing nor decreasing on all of $\mathbb{R}$. It is not one-to-one: for example, $(-1)^2 = 1^2$.

b. $y = x^2$ for $x \geq 0$ is strictly increasing, hence one-to-one.

c. $y = x^3$ is strictly increasing on $\mathbb{R}$, hence one-to-one.

d. $y = 2^{-x} = (1/2)^x$ is strictly decreasing, hence one-to-one.

e. $y = \ln x$ for $x > 0$ is strictly increasing, hence one-to-one.

6. Solve each demand equation for P:

a. $Q = 200 - 4P \;\Rightarrow\; 4P = 200 - Q \;\Rightarrow\; P = 50 - \dfrac{Q}{4}$.

b. $Q = 150 - 3P \;\Rightarrow\; 3P = 150 - Q \;\Rightarrow\; P = 50 - \dfrac{Q}{3}$.

c. $Q = 300 - 2P \;\Rightarrow\; 2P = 300 - Q \;\Rightarrow\; P = 150 - \dfrac{Q}{2}$.

d. $Q = 180 - 6P \;\Rightarrow\; 6P = 180 - Q \;\Rightarrow\; P = 30 - \dfrac{Q}{6}$.

e. $Q = 250 - 5P \;\Rightarrow\; 5P = 250 - Q \;\Rightarrow\; P = 50 - \dfrac{Q}{5}$.

7. Isolate P, keeping the nonnegative root since $P \geq 0$:

a. $Q = 100 - P^2 \;\Rightarrow\; P^2 = 100 - Q \;\Rightarrow\; P = \sqrt{100 - Q}$, for $0 \leq Q \leq 100$.

b. $Q = 64 - \sqrt{P} \;\Rightarrow\; \sqrt{P} = 64 - Q \;\Rightarrow\; P = (64 - Q)^2$, for $0 \leq Q \leq 64$.

c. $Q = 50 - 2\sqrt{P} \;\Rightarrow\; 2\sqrt{P} = 50 - Q \;\Rightarrow\; P = \dfrac{(50 - Q)^2}{4}$, for $0 \leq Q \leq 50$.

8.

a. $Q_D = 200 - 4P \;\Rightarrow\; 4P = 200 - Q_D \;\Rightarrow\; P = 50 - \dfrac{Q}{4}$.

b. Maximum price: set $Q = 0$, giving $P = 50$.

c. Maximum quantity: set $P = 0$, giving $Q = 200$.

9. For $F(K, L) = AK^aL^b$, scaling both inputs by c multiplies output by c^{a+b}:

a. $0.4 + 0.6 = 1$: constant returns to scale.

b. $0.7 + 0.5 = 1.2 > 1$: increasing returns to scale.

c. $0.3 + 0.5 = 0.8 < 1$: decreasing returns to scale.

d. $0.5 + 0.5 = 1$: constant returns to scale.

10.

a. Add willingness to pay at each quantity:

$$P_M(Q) = (50 - 2Q) + (30 - Q) = 80 - 3Q.$$

This uses both neighbors' demands, so it holds while both are positive, i.e. for $0 \leq Q \leq 25$.

b. Neighbor A: $50 - 2Q = 0 \Rightarrow Q = 25$. Neighbor B: $30 - Q = 0 \Rightarrow Q = 30$.

c. $P_M(10) = 80 - 3(10) = 50$.

11. Consumer 1's demand reaches zero at $p = 50$ ($100 - 2p = 0$), and consumer 2's demand reaches zero at $p = 50$ ($50 - p = 0$). Because both consumers exit at the same price, there is no kink: market demand is simply

$$Q_M = (100 - 2p) + (50 - p) = 150 - 3p, \qquad 0 \leq p \leq 50,$$

with $Q_M = 0$ for $p > 50$. Inverting gives the inverse market demand

$$p = 50 - \frac{Q_M}{3}, \qquad 0 \leq Q_M \leq 150.$$

12. Consumer 1's demand reaches zero at $P = 25$ ($50 - 2P = 0$); consumer 2's reaches zero at $P = 30$ ($30 - P = 0$). For $0 \leq P \leq 25$ both consumers are active; for $25 < P \leq 30$ only consumer 2 is active. Hence

$$Q_M = \begin{cases} 80 - 3P, & 0 \leq P \leq 25, \\ 30 - P, & 25 < P \leq 30, \\ 0, & P > 30. \end{cases}$$

The demand curve kinks at $(P, Q) = (25, 5)$. Inverting each segment,

$$P = \begin{cases} 30 - Q, & 0 \leq Q < 5, \\ \dfrac{80 - Q}{3}, & 5 \leq Q \leq 80. \end{cases}$$

13. Consumer 1's demand reaches zero at $p = 15$ ($30 - 2p = 0$); consumer 2's reaches zero at $p = 40$ ($40 - p = 0$). For $0 \le p \le 15$ both are active; for $15 < p \le 40$ only consumer 2 is active. Hence

$$Q_M = \begin{cases} 70 - 3p, & 0 \le p \le 15, \\ 40 - p, & 15 < p \le 40, \\ 0, & p > 40, \end{cases}$$

with the kink at $(p, Q) = (15, 25)$. The inverse market demand is

$$p = \begin{cases} 40 - Q, & 0 \le Q < 25, \\ \dfrac{70 - Q}{3}, & 25 \le Q \le 70. \end{cases}$$

14.

a. Hold Y fixed, so every pure-Y term behaves as a constant: $12X$ gives 12 and $-X^2$ gives $-2X$, so $MU_X = 12 - 2X$.

b. Likewise, $MU_Y = 18 - 3Y$.

c. $U(4, 6) = 12(4) + 18(6) - 4^2 - \frac{3}{2}(6^2) = 48 + 108 - 16 - 54 = 86$.

d. $MU_X(4, 6) = 12 - 2(4) = 4$; $MU_Y(4, 6) = 18 - 3(6) = 0$.

15.

a. Hold L fixed: $2K^2$ gives $4K$, $3KL$ gives $3L$, and $4L^2$ is constant, so $MP_K = 4K + 3L$.

b. Hold K fixed: $MP_L = 3K + 8L$.

c. $f(10, 20) = 2(10^2) + 3(10)(20) + 4(20^2) = 200 + 600 + 1600 = 2400$.

d. $MP_K(10, 20) = 4(10) + 3(20) = 100$; $MP_L(10, 20) = 3(10) + 8(20) = 190$.

16.

a. Hold y fixed: $5x^2$ gives $10x$, $3xy$ gives $3y$, and $2y^2$ is constant, so $MC_x = 10x + 3y$.

b. Hold x fixed: $MC_y = 3x + 4y$.

c. $C(6, 8) = 5(6^2) + 3(6)(8) + 2(8^2) = 180 + 144 + 128 = 452$.

d. $MC_x(6, 8) = 10(6) + 3(8) = 84$; $MC_y(6, 8) = 3(6) + 4(8) = 50$.

Chapter 8: Financial Mathematics and Summation

1. Simple interest: $FV = P(1 + rt)$.

a. $1000\Big[1 + 0.05(3)\Big] = 1000(1.15) = 1150$

b. $2000\Big[1 + 0.04(5)\Big] = 2000(1.20) = 2400$

c. $1500\Big[1 + 0.06(2)\Big] = 1500(1.12) = 1680$

2. Compound interest: $FV = P(1 + r)^n$.

a. $1000(1.05)^3 = 1000(1.157625) = 1157.63$

b. $2000(1.04)^5 = 2000(1.2166529) = 2433.31$

c. $1500(1.06)^2 = 1500(1.1236) = 1685.40$

d. $3000(1.07)^{10} = 3000(1.9671514) = 5901.45$

e. $5000(1.03)^8 = 5000(1.2667701) = 6333.85$

3. Present value: $PV = C/(1 + r)^t$.

a. $\dfrac{1000}{(1.05)^3} = \dfrac{1000}{1.157625} = 863.84$

b. $\dfrac{2000}{(1.04)^5} = \dfrac{2000}{1.2166529} = 1643.85$

c. $\dfrac{1500}{(1.06)^2} = \dfrac{1500}{1.1236} = 1334.99$

d. $\dfrac{5000}{(1.07)^{10}} = \dfrac{5000}{1.9671514} = 2541.75$

e. $\dfrac{3000}{(1.03)^8} = \dfrac{3000}{1.2667701} = 2368.25$

4. Discount each year's profit by the product of that year's and all earlier discount factors:

a. $PV = \dfrac{1000}{1.03} + \dfrac{1200}{(1.03)(1.04)} \approx 970.87 + 1120.24 = 2091.11.$

b. $PV = \dfrac{800}{1.02} + \dfrac{900}{(1.02)(1.03)} + \dfrac{1000}{(1.02)(1.03)(1.04)} \approx 784.31 + 856.65 + 915.23 = 2556.19.$

c. $PV = \dfrac{1500}{1.05} + \dfrac{2000}{(1.05)(1.06)} \approx 1428.57 + 1796.95 = 3225.52.$

d. $PV = \dfrac{2000}{1.04} + \dfrac{2000}{(1.04)(1.045)} + \dfrac{2000}{(1.04)(1.045)(1.05)} \approx 1923.08 + 1840.27 + 1752.63 = 5515.98.$

e. $PV = \dfrac{1000}{1.03} + \dfrac{1500}{(1.03)(1.035)} + \dfrac{2000}{(1.03)(1.035)(1.04)} + \dfrac{2500}{(1.03)(1.035)(1.04)(1.045)} \approx 970.87 + 1407.06 + 1803.93 + 2157.81 = 6339.67.$

5. Future value of an ordinary annuity: $FV = C\,\dfrac{(1+r)^n - 1}{r}$.

a. $100\,\dfrac{(1.05)^5 - 1}{0.05} = 100(5.5256314) = 552.56$

b. $200\,\dfrac{(1.04)^{10} - 1}{0.04} = 200(12.0061071) = 2401.22$

c. $300\,\dfrac{(1.06)^8 - 1}{0.06} = 300(9.8974679) = 2969.24$

d. $500\,\dfrac{(1.07)^{12} - 1}{0.07} = 500(17.8884514) = 8944.23$

6. Present value of an ordinary annuity: $PV = C\,\dfrac{1 - (1+r)^{-n}}{r}$.

a. $100\,\dfrac{1 - (1.05)^{-5}}{0.05} = 100(4.3294767) = 432.95$

b. $200\,\dfrac{1 - (1.04)^{-10}}{0.04} = 200(8.1108958) = 1622.18$

c. $300\,\dfrac{1 - (1.06)^{-8}}{0.06} = 300(6.2097938) = 1862.94$

d. $500\,\dfrac{1 - (1.07)^{-12}}{0.07} = 500(7.9426863) = 3971.34$

7. An annuity due is the ordinary annuity multiplied by $(1 + r)$, since every payment compounds one extra period:

a. $552.56(1.05) = 580.19$

b. $2401.22(1.04) = 2497.27$

8. Perpetuity: $PV = C/r$.

a. $\dfrac{100}{0.05} = 2000$

b. $\dfrac{200}{0.04} = 5000$

c. $\dfrac{50}{0.025} = 2000$

9. Perpetuity due: one payment today plus an ordinary perpetuity, $PV = C + C/r$.

a. $100 + \dfrac{100}{0.05} = 2100$

b. $200 + \dfrac{200}{0.04} = 5200$

c. $50 + \dfrac{50}{0.025} = 2050$

10.

a. $\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5 = 15$

b. $\sum_{i=1}^{4}(2i + 1) = 3 + 5 + 7 + 9 = 24$

c. $\sum_{i=1}^{3} i^2 = 1 + 4 + 9 = 14$

d. $\sum_{i=1}^{10} 5 = 10 \cdot 5 = 50$

e. $\sum_{i=1}^{6} 3i = 3(1 + 2 + 3 + 4 + 5 + 6) = 3(21) = 63$

11.

a. $\sum_{i=1}^{100} 4 = 100 \cdot 4 = 400$

b. $\sum_{i=1}^{20}(3i + 2) = 3\frac{20\cdot 21}{2} + 2(20) = 630 + 40 = 670$

c. $\sum_{i=1}^{10}(i^2 + 3i) = \frac{10\cdot 11\cdot 21}{6} + 3\frac{10\cdot 11}{2} = 385 + 165 = 550$

12. Use $a\dfrac{1 - r^n}{1 - r}$:

a. $1 + 2 + 4 + 8 + 16 = 1 \cdot \dfrac{1 - 2^5}{1 - 2} = 31$

b. $3 + 6 + 12 + 24 = 3 \cdot \dfrac{1 - 2^4}{1 - 2} = 45$

c. $100 + 50 + 25 + 12.5 = 100 \cdot \dfrac{1 - (0.5)^4}{1 - 0.5} = 187.5$

d. $5 + 5(1.05) + 5(1.05)^2 + 5(1.05)^3 = 5 \cdot \dfrac{1 - (1.05)^4}{1 - 1.05} \approx 5(4.310125) = 21.55$

13. Continuous compounding: $FV = Pe^{rt}$.

a. $1000e^{0.05 \cdot 10} = 1000e^{0.5} \approx 1000(1.6487213) = 1648.72$

b. $2000e^{0.04 \cdot 5} = 2000e^{0.2} \approx 2000(1.2214028) = 2442.81$

c. $5000e^{0.06 \cdot 3} = 5000e^{0.18} \approx 5000(1.1972174) = 5986.09$

14. $NPV = -5000 + \dfrac{2000}{1+r} + \dfrac{2000}{(1+r)^2} + \dfrac{2000}{(1+r)^3}$:

a. At $r = 5\%$:

$$NPV = -5000 + \frac{2000}{1.05} + \frac{2000}{(1.05)^2} + \frac{2000}{(1.05)^3} \approx -5000 + 1904.76 + 1814.06 + 1727.68 = 446.50.$$

Since $NPV > 0$, the project should be undertaken.

b. At $r = 10\%$:

$$NPV = -5000 + \frac{2000}{1.10} + \frac{2000}{(1.10)^2} + \frac{2000}{(1.10)^3} \approx -5000 + 1818.18 + 1652.89 + 1502.63 = -26.30.$$

Since $NPV < 0$, the project should be rejected. A higher discount rate makes the future payments worth less today.